San José State University

Math 161A: Applied Probability & Statistics

Joint distributions

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Section 5.1 Jointly Distributed Random Variables

Section 5.2 Expected Value (Covariance and Correlation not covered)

Introduction

So far we have considered the distribution of only a single random variable, discrete or continuous.

When two or more random variables are defined on the same sample space, we can describe their **joint distribution**.

Example 0.1 (Toss two fair dice)**.** Let X denote their sum and Y the absolute value of their difference, which are two discrete random variables. We can find their individual distributions easily:

x	2	3	$\cdots$	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\cdots$	$\frac{1}{36}$

y	0	1	$\cdots$	5
$P(Y=y)$	$\frac{6}{36}$	$\frac{10}{36}$	$\cdots$	$\frac{2}{36}$

Now consider X, Y together as a pair (X, Y), or a vectored-valued function.

Questions:

- Can (X, Y) attain all the $66 = 11 \times 6$ pairs?

$$\{(x, y) \mid 2 \leq x \leq 12,\ 0 \leq y \leq 5\}$$

  If not all, identify the subset of feasible pairs.

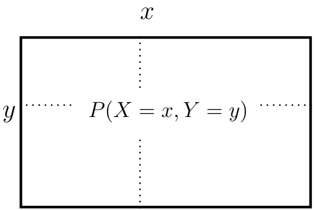
- What are the corresponding probabilities for (X, Y) to take those (feasible) pairs as values?

Answering the above two questions together is equivalent to specifying the **joint probability distribution of** (X, Y) in terms of range and frequency.

The joint pmf

Def 0.1. Let X, Y be two discrete random variables defined on the same sample space. We define their joint pmf as a function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x,y) = \begin{cases} P(X=x, Y=y), & \text{for all feasible pairs } (x,y) \\ 0, & \text{otherwise} \end{cases}$$



Example 0.2. Find the joint pmf of X, Y in the previous example.

y \ x	2	3	4	5	6	7	8	9	10	11	12
0											
1											
2											
3											
4											
5											

y \ x	2	3	4	5	6	7	8	9	10	11	12
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$
1											
2											
3											
4											
5											

y \ x	2	3	4	5	6	7	8	9	10	11	12
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$
1		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$	
2											
3											
4											
5											

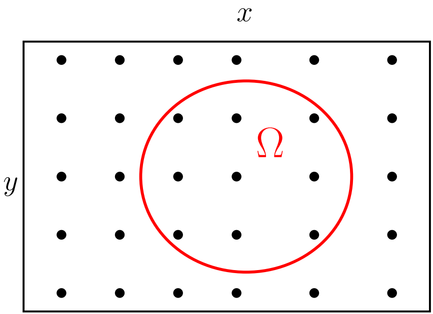
y \ x	2	3	4	5	6	7	8	9	10	11	12
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$
1		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$	
2			$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		
3				$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$			
4					$\frac{2}{36}$		$\frac{2}{36}$				
5						$\frac{2}{36}$					

Remark. Any joint pmf $f(x, y) : \mathbb{R}^2 \mapsto \mathbb{R}$ must satisfy (and vice versa)

- $f(x, y) \geq 0$ for all $x, y \in \mathbb{R}$
- $f(x, y) > 0$ for finitely or countably many pairs (x, y);
- $\sum_x \sum_y f(x, y) = 1$.

Theorem 0.1. Let X, Y be two discrete random variables with joint pmf $f(x, y)$. Then for any region $\Omega \subset \mathbb{R}^2$,

$$P((X,Y) \in \Omega) = \sum_{(x,y)\in\Omega} f(x,y)$$



Example 0.3 (Toss 2 fair dice, cont'd)**.** Find the following probabilities:

- $P(X \leq 4, Y \leq 2) = \frac{6}{36}$ (sum of top-left 3×3 block of joint pmf table on slide 9)
- $P(X \leq 5) = \frac{10}{36}$ (sum of first four columns of joint pmf table)
- $P(X \geq 11, Y \leq 2) = \frac{3}{36}$ (sum of top-right 3×2 block of joint pmf table)
- $P(Y \leq 1) = \frac{16}{36}$ (sum of first two rows of joint pmf table)

From joint to marginal

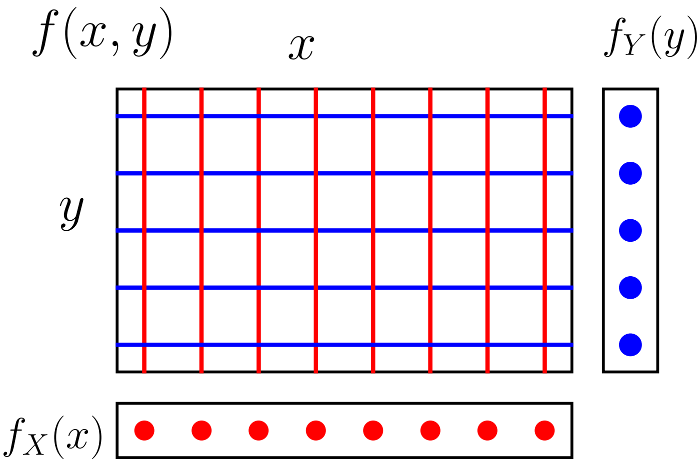
Def 0.2. For any two discrete random variables X, Y that have a joint distribution, we call their individual pmfs $f_X(x), f_Y(y)$ the **marginal pmfs**.

Proposition 0.2. Let $f(x, y)$ be the joint pmf for X, Y. Then

$$f_X(x) = \sum_y f(x, y), \quad \text{and} \quad f_Y(y) = \sum_x f(x, y).$$

Proof. This is just the Law of Total Probability:

$$\underbrace{P(X = x)}_{f_X(x)} = \sum_y \underbrace{P(X = x, Y = y)}_{f(x,y)}.$$


$f(x, y)$
x
$f_Y(y)$
y
$f_X(x)$

$y \backslash x$	2	3	4	5	6	7	8	9	10	11	12	$f_Y(y$
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$	$\frac{6}{36}$
1		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{10}{36}$
2			$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$			$\frac{8}{36}$
3				$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$				$\frac{6}{36}$
4					$\frac{2}{36}$		$\frac{2}{36}$					$\frac{4}{36}$
5						$\frac{2}{36}$						$\frac{2}{36}$
$f_X(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	

Conditional pmfs

Consider the following question:

Example 0.4 (Toss 2 fair dice)**.** Suppose we are told that the sum is $X = 6$. What is the (conditional) distribution of Y?

Answer:

y	0	2	4
$P(Y = y \mid X = 6)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

Def 0.3. Let X, Y be two discrete random variables with joint pmf $f(x, y)$. The conditional pmf of Y given $X = x$ (with $f_X(x) \neq 0$)) is defined as

$$f(\underbrace{y}_{\text{variable}} \mid \underbrace{x}_{\text{fixed}}) = \frac{f(x,y)}{f_X(x)}, \quad \text{for all feasible } y$$

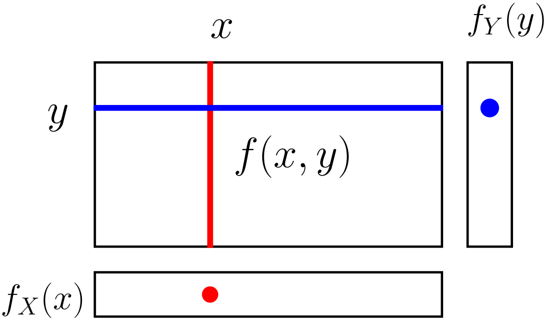
Remarks:

(1) This definition is just based on the conditional probability of events:

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)}.$$

(2) For each fixed value x of X, there is a separate conditional distribution for Y at x (x may be regarded as a location parameter).

Joint distributions



$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

$$f(y \mid x) = \frac{f(x,y)}{f_X(x)}$$

Table 1: Conditional pmfs of Y given $X = x$

y \ x	2	3	4	5	6	7	8	9	10	11	12
0	1		$\frac{1}{3}$		$\frac{1}{5}$		$\frac{1}{5}$		$\frac{1}{3}$		1
1		1		$\frac{1}{2}$		$\frac{1}{3}$		$\frac{1}{2}$		1	
2			$\frac{2}{3}$		$\frac{2}{5}$		$\frac{2}{5}$		$\frac{2}{3}$		
3				$\frac{1}{2}$		$\frac{1}{3}$		$\frac{1}{2}$			
4					$\frac{2}{5}$		$\frac{2}{5}$				
5						$\frac{1}{3}$					

Table 2: Conditional pmfs of X given $Y = y$

y \ x	2	3	4	5	6	7	8	9	10	11	12
0	$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$		$\frac{1}{6}$
1		$\frac{1}{5}$		$\frac{1}{5}$		$\frac{1}{5}$		$\frac{1}{5}$		$\frac{1}{5}$	
2			$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{4}$		$\frac{1}{4}$		
3				$\frac{1}{3}$		$\frac{1}{3}$		$\frac{1}{3}$			
4					$\frac{1}{2}$		$\frac{1}{2}$				
5						1					

Example 0.5 (Toss two fair dice)**.** Find the following conditional distributions:

- Y given $X = 4$:

y	0	2
$f(y \mid x = 4)$	$\frac{1}{3}$	$\frac{2}{3}$

- X given $Y = 3$:

x	5	7	9
$f(x \mid y = 3)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

- X given $Y = 0$:

x	2	4	6	8	10	12
$f(x \mid y = 0)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Independence

Def 0.4. Two discrete random variables X, Y are **independent** if

$$f(x,y) = f_X(x) f_Y(y), \quad \text{for all } x, y$$

Remark. This is equivalent to

$$P(X = x, Y = y) = P(X = x) P(Y = Y), \quad \text{for all } x, y.$$

Proposition 0.3. Two discrete random variables X, Y are independent if all conditional distributions of Y are identical to its marginal distribution:

$$f(y \mid x) = f_Y(y), \quad \text{for all } x, y$$

Example 0.6 (Toss 2 fair dice). Determine if X (sum) and Y (absolute difference) are independent.

$y \backslash x$	2	3	4	5	6	7	8	9	10	11	12	$f_Y(y)$
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$	$\frac{6}{36}$
1		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{10}{36}$
2			$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$			$\frac{8}{36}$
3				$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$				$\frac{6}{36}$
4					$\frac{2}{36}$		$\frac{2}{36}$					$\frac{4}{36}$
5						$\frac{2}{36}$						$\frac{2}{36}$
$f_X(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	

Example 0.7. Are the random variables X, Y independent?

$y \backslash x$	0	1	2
-1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$
1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

Answer: Yes, because $f(x, y) = f_X(x)f_Y(y)$ for all x, y. An alternative way is to compare the conditional distributions $f(y \mid x)$ for each x (if they are all identical, then X, Y are independent).

$y \backslash x$	0	1	2	$f_Y(y)$
-1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$
1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
$f_X(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	

$y \backslash x$	0	1	2
-1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

Expected value of $g(X, Y)$

Consider the following question:
Given two discrete random variables X, Y with a joint distribution, what are the expected values of random variables like $X + Y, XY, |X - Y|$?

Theorem 0.4. Let X, Y be two discrete random variables with a joint pmf $f(x, y)$. Then for any function $g(X, Y)$,

$$\mathrm{E}(g(X,Y)) = \sum_x \sum_y g(x,y) f(x,y).$$

Example 0.8. Consider the random variables X, Y in the preceding example, compute:

$y \backslash x$	0	1	2	$f_Y(y)$
-1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$
1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
$f_X(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	

$\mathrm{E}(X+Y) =$

$\mathrm{E}(XY) =$

$\mathrm{E}(|X-Y|) =$

Two continuous random variables (skipped)

The joint distribution of two continuous random variables is also described by a 2D function $f(x, y)$, called **joint pdf**. However, since probability calculations will involve multiple integration, this topic is not covered.

Example 0.9. Consider the game of throwing a dart toward a unit disk and let X, Y be the coordinates of the landing point (assuming it is always within the disk). Individually, X, Y both range from -1 to 1, but the pair (X, Y) does not attain every point in the square.