San José State University

Math 161A: Applied Probability & Statistics

Lecture 1: Probability Basics

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Section 2.1 Sample Spaces and Events

Section 2.2 Axioms, Interpretations, and Properties of Probability

Introduction

To model a random phenomenon (such as flipping a coin, rolling a die), we need to specify the following components:

- **Sample space**
- **Events**
- **Probability**

Collectively, they define a **probability space**.

We'll go through the above concepts one by one.

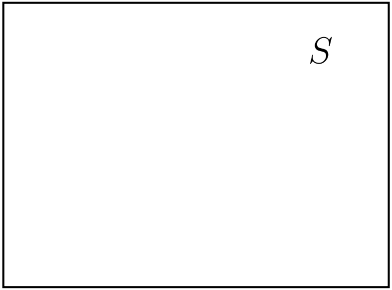
A little bit set terminology and notation:

- A **set** is an unordered collection of objects: $A = \{1, 2, 3\} = \{3, 1, 2\}$, $B = (1, 3) = \{x \mid 1 < x < 3\}$, $C = [0, \infty) = \{x \mid x \geq 0\}$
- **Notation**: To indicate an object is in or not in the set: $1 \in A$, $4 \notin A$
- A **subset** is a subcollection of the objects: $\{1\} \subset A \subset C$
- The **size** (cardinality) of a set is the number of objects in it: $|A| = 3$
- **Special sets**: $\emptyset$ (empty set), $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, $\mathbb{R} = (-\infty, \infty)$

Sample space

Def 0.1. The set of all possible outcomes of a random phenomenon is called the sample space for that experiment.

- We denote a sample space by S (some books use Ω instead).
- A sample space is typically represented by a rectangle, and the outcomes of the sample space are denoted by points within the rectangle.



Example 0.1. Write down the sample space of each of the following experiments:

- Tossing a coin: $S = \{H,\ T\}$.
- Rolling a die: $S = \{1, 2, 3, 4, 5, 6\}$.
- Drawing a card from an ordinary deck of 52: $S = \{\text{All 52 cards}\}$.

Example 0.2. Write down the sample space of each of the following experiments:

- Throw a coin twice:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}.$$

- Roll two dice:

$$\begin{aligned} S &= \{(1,1), \ldots, (1,6), (2,1), (2,2), \ldots, (6,6)\} \longleftarrow \text{by enumeration} \\ &= \{(i,j) \mid 1 \le i \le 6, 1 \le j \le 6\} \quad \longleftarrow \text{by formula} \end{aligned}$$

- Throw a coin repeatedly until a head first appears:

$$S = \{H,\ TH,\ TTH,\ TTTH, \ldots\}$$

The sample spaces in the previous example are **discrete** sets, which are countable. That is, the number of objects in the set must be

- finite (e.g., $\{1, 2, \ldots, 6\}$), or
- countably infinite: There is a 1-to-1 correspondence to the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

For example, the set of all integers $\mathbb{Z}$ is countable:

1	2	3	4	5	6	7	8	9	10	$\cdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$
0	1	-1	2	-2	3	-3	4	-4	5	$\cdots$

In contrast, the set of all real numbers $\mathbb{R}$ is uncountable.

In the following example, the sample spaces are **continuous intervals** (which are uncountable).

Example 0.3.

- Life time of a new light bulb. The sample space is an interval $S = (0, \infty)$.
- Waiting time (in minutes) to talk to a customer service representative: $S = (0, \infty)$
- Throw a dart to a unit disk and measure its distance to center: $S = [0, 1]$

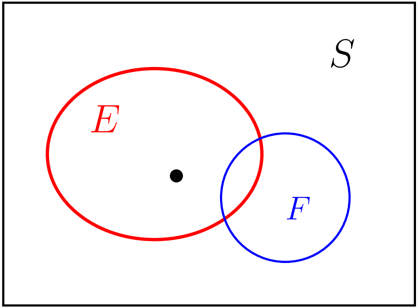
Events

Consider the following probability questions about events:

- (**Toss two fair dice**) What is the probability of getting a sum of 8?
- (**Toss two fair dice**) What is the probability of getting two even numbers?
- (**Toss two fair dice**) What is the probability of getting two identical numbers?
- (**Toss a fair coin repeatedly until a head first appears**) What is the probability that at most 3 tails are observed?

Def 0.2. Mathematically, an event is just a subset E of outcomes in the sample space S.

- In particular, $S, \emptyset$ are events.
- We say an event E **occurred** if the actual outcome of the experiment lies in E.
- It is called a **simple** event if it contains only one outcome. Otherwise, it is called a **compound** event.



(Event E occurred, while F did not)

Example 0.4 (Roll a single die). The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. The following are events:

- $A = \{1\} = \{\text{The smallest number}\}$ $\longleftarrow$ simple event
- $B = \{6\} = \{\text{The largest number}\}$ $\longleftarrow$ simple event
- $C = \{2, 4, 6\} = \{\text{An even number}\}$ $\longleftarrow$ compound event
- $D = \{1, 3, 5\} = \{\text{An odd number}\}$ $\longleftarrow$ compound event

If an outcome of 1 was observed when performing the experiment, then which of the above events occurred (and which of them did not occur)?

Example 0.5 (Throw two dice)**.** The sample space is

$$S = \{(1,1),(1,2),\ldots,(6,6)\} = \{(i,j) \mid 1 \le i,j \le 6\}.$$

The following are events:

$$\begin{aligned} A &= \{\text{Sum equals } 6\} \\ &= \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \\ B &= \{\text{Two identical numbers}\} \\ &= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\} \\ C &= \{\text{Two even numbers}\} \\ &= \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}. \end{aligned}$$

Example 0.6. Consider the experiment where you repeatedly toss a coin until you see the first head. The following is an event:

$$\begin{aligned} E &= \{\text{At most 4 tails are obtained}\} \\ &= \{H, TH, TTH, TTTH, TTTTH\} \end{aligned}$$

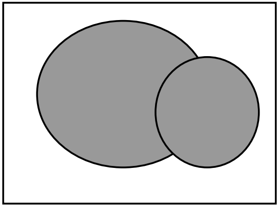
Event operations

Def 0.3. Let $A, B \subseteq S$ be two events. We define

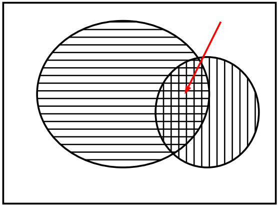
- **Complement** A^c: set of all outcomes not in A
- **Union** $A \cup B$: set of all outcomes in A or B (or both)
- **Intersection** $A \cap B$: set of all outcomes in both A and B
- **Difference** $A - B = A \cap B^c$: set of all outcomes in A and not in B

They can be represented by the so-called Venn diagrams (see next slide).

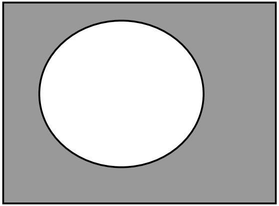
$A \cup B$



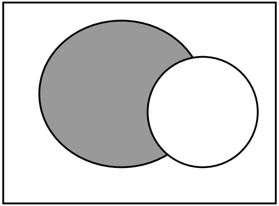
$A \cap B$



A^c



$A - B$



Example 0.7 (Throw two dice)**.** Let

- $A = \{\text{Sum equals 6}\}$
- $B = \{\text{Two identical numbers}\}$
- $C = \{\text{Two even numbers}\}$

Compute $|C|, A \cap B, A \cup B, B^c, A - C$

Two useful set laws:

- **Distributive law**:

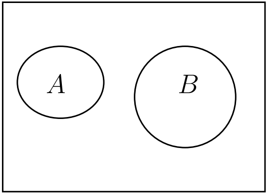
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- **De Morgan's Laws**

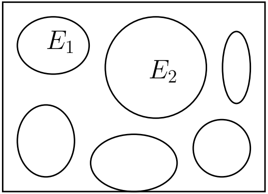
$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c$$

Disjoint events

Def 0.4. Two events A, B are said to be **disjoint**, or **mutually exclusive**, if their intersection is empty: $A \cap B = \emptyset$.



A sequence of events $E_1, E_2, \ldots$ are said to be **pairwise disjoint**, or **mutually exclusive**, if $E_i \cap E_j = \emptyset$ for all $i \neq j$.



Example 0.8 (Toss two fair dice)**.** Are the following two events disjoint?

- $A = \{\text{Sum equals 7}\}$.
- $B = \{\text{Two identical numbers}\}$.

Probability

Intuitively, probability is a number $P(E)$ describing the chance of an event E occurring.

The larger the probability, the more likely for the event to occur.

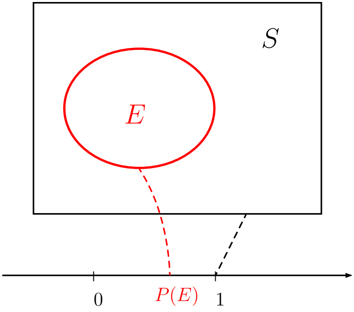
And it needs to satisfy certain conditions in order to be valid/meaningful.

Below is the formal definition of probability.

Def 0.5. Probability is a function defined on the space of events that satisfies the following Kolmogorov Axioms of Probability:

1. $P(E) \geq 0$ for any $E \subseteq S$.
2. $P(S) = 1$.
3. For any infinite sequence of pairwise disjoint events $E_1, E_2, \ldots$,

$$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$



Theorem 0.1. The Axioms of Probability imply the following are true:[1]

- $P(\emptyset) = 0$.
- If $E_1, E_2, \ldots, E_k \subset S$ are pairwise disjoint, then

$$P\left(\cup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} P(E_i)$$

- $P(E^c) = 1 - P(E)$, from which we obtain that $P(E) \leq 1$.
- $P(B - A) = P(B) - P(B \cap A)$: If $A \subseteq B$, then it simplifies to $P(B - A) = P(B) - P(A)$.

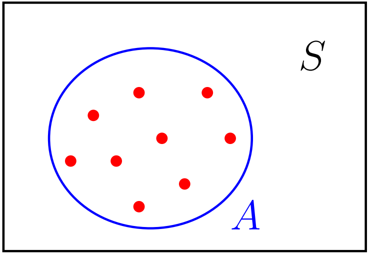
[1] This is why we did not include these properties in the definition of probability.

Countable sample spaces

The following property implies that, to define the probability function over a **countable** sample space, it suffices to specify only the probabilities of simple events.

Theorem 0.2. If the sample space S is countable, then for any event $A \subseteq S$,

$$P(A) = \sum_{a \in A} P(\{a\}).$$



Example 0.9 (Fair coin model)**.** Let $S = \{H, T\}$ with $P(\{H\}) = P(\{T\}) = .5$.

Example 0.10 (Biased coin model)**.** Let $S = \{H, T\}$ with $P(\{H\}) = .55, P(\{T\}) = .45$.

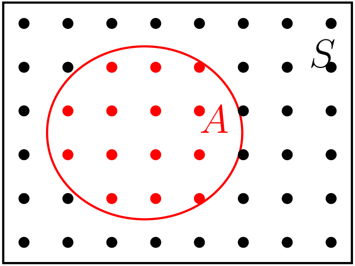
Example 0.11 (Fair die model)**.** Let $S = \{1, 2, \ldots, 6\}$ with $P(\{1\}) = P(\{2\}) = \cdots = P(\{6\}) = \frac{1}{6}$. The probability of getting an even number is

$$P(\{\text{An even number}\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

Finite sample spaces with equally likely outcomes

Theorem 0.3. If $|S| < \infty$ (i.e., S is a finite set) and all the outcomes are equally likely to occur, then for any event $A \subseteq S$,

$$P(A) = \frac{|A|}{|S|} = \frac{\#\text{ outcomes in } A}{\#\text{ outcomes in } S}.$$



Proof. By the preceding theorem,

$$P(A) = \sum_{a \in A} P(\{a\}) = \sum_{a \in A} \frac{1}{|S|} = \frac{1}{|S|} \cdot |A| = \frac{|A|}{|S|}.$$

Joke: What is a probability to meet a dinosaur?

A: What is a probability to meet a dinosaur on the street?

B: Well, 50x50!

A: How, why???

B: You either meet it or not!

So, i met it!

Example 0.12 (Throw a fair die). Find the following probabilities:

$P(\{\text{An even number})\}) =$

$P(\{\text{At least } 5\}) =$

$P(\{\text{Not a } 3\}) =$

Example 0.13 (Throw two fair dice)**.** Find the following probabilities:

$P(\{\text{Sum equals } 6\}) =$

$P(\{\text{Two identical numbers}\}) =$

$P(\{\text{Both even}\}) =$

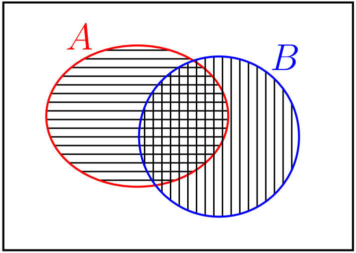
Example 0.14 (Toss a fair coin 5 times). What is the probability of getting at least one head?

Inclusive-exclusive formula (2 events)

Theorem 0.4. For any $A, B \subseteq S$,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

In particular, if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.



Proof. By additivity for mutually exclusive events,

$$\begin{aligned} P(A \cup B) &= P(A - B) + P(A \cap B) + P(B - A) \\ &= P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(B \cap A) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

Example 0.15. In a large discrete math class, 55% of the students have a major in math, and 35% of the class have a major in CS. Among the two groups of students combined, 5% of them are dual majors (in math and CS). What is the probability that a randomly selected student from the class majors in

(a) at least one of math and CS,

(b) one and only one of math and CS,

(c) neither math nor CS?

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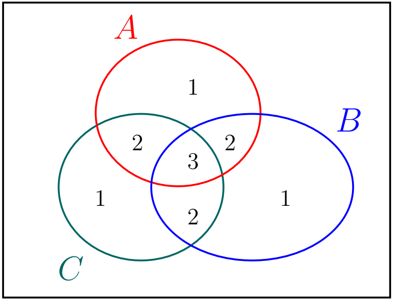
Inclusive-exclusive formula (3 events)

Theorem 0.5. For any three events $A, B, C \subseteq S$, we have

$$P(A \cup B \cup C)$$
$$= P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C).$$



Example 0.16 (Select an integer from $\{1, \ldots, 100\}$ at random)**.** What is the probability that it is divisible by at least one of the three prime numbers 2, 3, 5? (Answer: .74)

Summary

We first introduced the concept of a **probability space** associated to a random phenomenon, which consists of the following:

- **Sample space** S (set of all possible outcomes)
- **Events** $E \subseteq S$ (subsets of outcomes, often with a common trait)
- **Probability** (chance that an event occurs): a mapping from events to numbers, $P : E \subseteq S \mapsto P(E) \in \mathbb{R}$, that satisfies the three Axioms of Probability

  1. $P(E) \geq 0$ for any $E \subseteq S$.

2. $P(S) = 1$.

3. If an infinite sequence of events $E_1, E_2, \ldots$ are pairwise disjoint, then

$$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

The Axioms imply more properties for the probability function:

- $P(\emptyset) = 0$.
- If $E_1, E_2, \ldots, E_k$ are pairwise disjoint, then

$$P\left(\cup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} P(E_i)$$

- $P(E^c) = 1 - P(E)$, from which we obtain that $P(E) \leq 1$.
- If $A \subseteq B$, then $P(A) \leq P(B) \longleftarrow$ This is due to the property $P(B - A) = P(B) - P(A \cap B)$
- Inclusive-exclusive formula for any two events $A, B \subseteq S$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Inclusive-exclusive formula for any three events $A, B, C \subseteq S$:

$$\begin{aligned} P(A \cup B \cup C) = & \, P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C). \end{aligned}$$

Lastly, there are two special settings:

- If the sample space S is countable, then for any event $A \subseteq S$,

$$P(A) = \sum_{a \in A} P(\{a\}).$$

- If the sample space is finite and all the outcomes are equally likely to occur, then for any event $A \subseteq S$,

$$P(A) = \frac{|A|}{|S|} = \frac{\#\text{ outcomes in } A}{\#\text{ outcomes in } S}.$$