Worksheet 13: hypothesis testing

Example 0.103. In the brown egg problem, suppose the true population standard deviation $\sigma = 2$ grams. A person decides to use the following decision rule (for a sample of size $n = 12$, i.e., a carton of eggs)

$$|\bar{x} - 65| > 1$$

to conduct the two-sided test

$$H_0 : \mu = 65 \qquad \text{vs} \quad H_1 : \mu \neq 65.$$

What is the probability α of making a type-I error? (Answer: 0.0836)

Example 0.104. (cont'd) Consider two different decision rules:

- $|\bar{x} - 65| > 0.5$
- $|\bar{x} - 65| > 2$

for conducting the same two-sided test. Verify that the corresponding probabilities of making a type-I error are $0.3844, 0.0006$, respectively.

Example 0.105. Compute the probability of making a type-I error for the one-sided test $H_1 : \mu < 65$ with each of the following decision rules

- $\bar{x} < 65 - 0.5 = 64.5$
- $\bar{x} < 65 - 1 = 64$
- $\bar{x} < 65 - 2 = 63$

(Answers: $0.1922,\ 0.0418,\ 0.0003$)

Example 0.106. Consider the two-sided test in the eggs example:

$$H_0 : \mu = 65 \qquad \text{vs} \quad H_1 : \mu \neq 65.$$

and the following decision rule:

$$|\bar{x} - 65| > c$$

Find the probability of making a type-II error when $\mu = 64$ for $c = 1/2, 1, 2$. (Answer: $\beta = P(|\bar{X} - 65| < c \mid \mu = 64) = 0.1875, 0.4997, 0.9582$)

Example 0.107. Assume the setting of the brown eggs example (with known $\sigma = 2$, but sample size n TBD). Consider the following one-sided test

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu < 65$$

with decision rule

$$\bar{x} < 65 - c$$

Choose n, c so that the test has level 5% and power 80% (at $\mu = 64$).

Answer: $c = z_\alpha \frac{\sigma}{\sqrt{n}} = 0.658,\ n = \left(\frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'}\right)^2 = 25$

Example 0.108. Redo the preceding example but instead for a two-sided test

$$H_0 : \mu = 65 \quad \text{vs} \quad H_a : \mu \neq 65$$

with corresponding decision rule

$$|\bar{x} - 65| > c$$

Answer: $c = z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 0.693,\ n \approx \left(\frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu'}\right)^2 = 32$

Example 0.109. In the brown eggs example, suppose we observed $\bar{x} = 63.8$.

- $H_1 : \mu \neq 65$: The more contradictory values are $\bar{x} < 63.8$ and $\bar{x} > 66.2$ (mirror point). Thus, for a 2-sided test,

$$\begin{aligned}\text{pval}(63.8) &= 2 \cdot P(\bar{X} \leq 63.8 \mid H_0 \text{ true}) \\ &= 2 \cdot P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{63.8 - 65}{2/\sqrt{12}} \mid \mu = 65\right) \\ &= 2 \cdot P(Z \leq -2.08) = 2 \cdot .019 = .038\end{aligned}$$

- $H_1 : \mu \neq 65$: The more contradictory values are only $\bar{x} < 63.8$. In this case, the p-value is

$$\text{pval}(63.8) = P(\bar{X} \leq 63.8 \mid H_0 \text{ true}) = .019$$

Example 0.110. In the previous example, what is your conclusion if $\alpha = 5\%$? 1%?

Example 0.111. Consider the egg-weight example again. Conduct the following test at level 95%

$$H_0 : \mu = 65 \qquad vs \qquad H_1 : \mu \neq 65$$

for a specific sample of 12 eggs with $\bar{x} = 64$ and $s^2 = 4.69$. Conduct the test at level $\alpha = .05$. What is the p-value of the sample?

(Answer: $|\frac{\bar{x}-65}{s/\sqrt{n}}| = 1.6 < t_{\alpha/2,n-1} = 2.201$, thus failing to reject the null. p-value=.138)

Example 0.112 (Continuation of previous example)**.** Conduct the following test at level 5%:

$$H_0 : \sigma^2 = 2^2 \qquad vs \qquad H_1 : \sigma^2 > 2^2$$

What is the p-value?

(Answer: $\frac{(n-1)s^2}{\sigma_0^2} = 12.9 < \chi^2_{\alpha,n-1} = 19.7$, thus failing to reject the null. $p$val=.3)