San José State University

Math 263: Stochastic Processes

Classification of States

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This lecture is based on the following textbook sections:

- Sections 4.3, 4.4a

Outline of the presentation

- Recurrent / transient states
- Periodic states

HW3: Assigned in Canvas

Def 0.1. Let $\{X_n, n \geq 0\}$ be a Markov chain with state space S. State j is said to be **accessible** from state i if

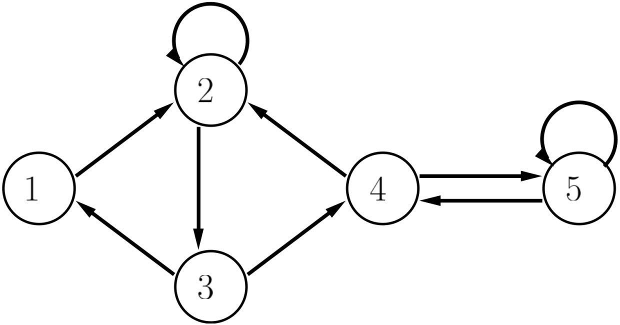
$$p_{ij}^{(n)} > 0 \quad \text{for some } n \geq 0.$$

We say that two states i, j **communicate** if they are accessible from each other, i.e.,

$$p_{ij}^{(n)} > 0,\ p_{ji}^{(m)} > 0 \quad \text{for some } n, m \geq 0.$$

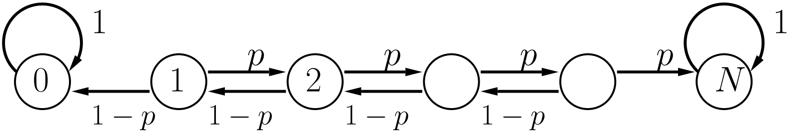
and we write $i \longleftrightarrow j$.

Example 0.1. Are the states in the following Markov chain accessible from each other?



Example 0.2. Consider the random walk with two absorbing barrers $0, N$:

- State 0 is accessible from any state $0 < i < N$, but not the other way. Thus, they do not communicate.
- States 1 and $N-1$ communicate.



Theorem 0.1. Communication is an equivalence relation (among the states of a Markov Chain):

- **Reflexivity**: for any state i, we have $i \longleftrightarrow i$;
- **Symmetry**: if $i \longleftrightarrow j$, then $j \longleftrightarrow i$
- **Transitivity**: if $i \longleftrightarrow j$ and $j \longleftrightarrow k$, then $i \longleftrightarrow k$.

This indicates that communication (as an equivalence relation) partitions the state space into disjoint equivalence classes.

Proof. We verify each property:

- For any state i,

  $$p_{ii}^{(0)} = P(X_0 = i \mid X_0 = i) = 1 > 0.$$

  This shows that $i \longleftrightarrow i$.

- Obvious.

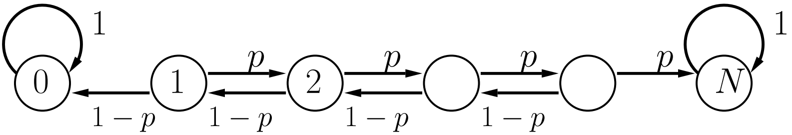
- We first show that k is accessible from i (the other direction can be proved in the same way). Suppose that

$$p_{ij}^{(n)} > 0, \quad p_{jk}^{(m)} > 0$$

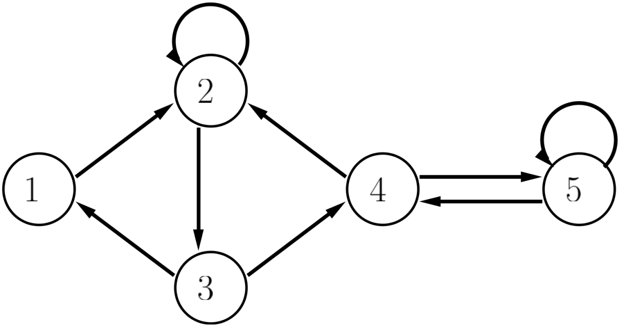
for some $n, m \geq 0$. Then

$$p_{ik}^{(n+m)} = \sum_{\ell} p_{i\ell}^{(n)} p_{\ell k}^{(m)} \geq p_{ij}^{(n)} p_{jk}^{(m)} > 0.$$

Example 0.3. How many communicating classes does the following chain have?

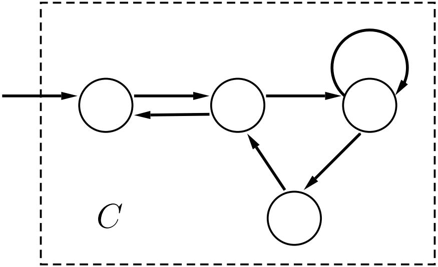


Def 0.2. A Markov chain is said to be **irreducible** if it consists of only 1 communicating class, that is, all states communicate with each other.

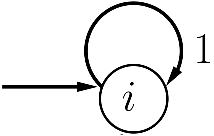


Def 0.3. A communicating class C of states is said to be **closed** if it is not possible to leave that class (once entering it), that is,

$$P_{ij} = 0, \qquad \text{whenever} \quad i \in C \text{ and } j \notin C.$$



Def 0.4. A state i is said to be an **absorbing state** if $\{i\}$ is a closed class.



For any state i, we denote by f_{ii} the probability that starting in state i, the process will ever reenter i:

$$\begin{aligned} f_{ii} &= P(X_n = i \text{ for some finite } n \mid X_0 = i) \\ &= P\left(\bigcup_{n=1}^{\infty} \{X_n = i\} \,\middle|\, X_0 = i \right) \end{aligned}$$

Def 0.5. State i is said to be

- **recurrent** if $f_{ii} = 1$, or
- **transient** if $f_{ii} < 1$.

Remark.

(1) Any recurrent state must be visited infinitely often when the chain originates from it;

(2) Starting in a transient state i, the number of times the process will reenter the state has a geom$(1-f_{ii})$ distribution with finite mean $\frac{1}{1-f_{ii}}$.

Therefore, **state i is recurrent if and only if the expected number of time periods the process is in that state is infinity**.

Theorem 0.2. State i is recurrent if

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty,$$

or transient if

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty.$$

Proof. For a fixed state i, let $I_n = 1_{X_n=i}$ for $n = 1, 2, \ldots$. Then the number of time periods the process is in state i is

$$N = \sum_{n=1}^{\infty} I_n.$$

It follows that

$$\begin{aligned}
\mathrm{E}(N \mid X_0 = i) & \sum_{n=1}^{\infty} \mathrm{E}(I_n \mid X_0 = i) \\
&= \sum_{n=1}^{\infty} P(X_n = i \mid X_0 = i) \\
&= \sum_{n=1}^{\infty} p_{ii}^{(n)}.
\end{aligned}$$

Combining this and the remark completes the proof. □

Corollary 0.3. If state i is recurrent and $i \longleftrightarrow j$, then j is also recurrent.

Proof. Suppose that

$$p_{ij}^{(n)} > 0 \quad \text{for some } n \geq 0$$

and

$$p_{ji}^{(m)} > 0 \quad \text{for some } m \geq 0.$$

Then for any $k \geq 1$,

$$p_{jj}^{(n+m+k)} \geq p_{ji}^{(m)} p_{ii}^{(k)} p_{ij}^{(n)}.$$

Taking the sum over k yields that

$$\sum_k p_{jj}^{(n+m+k)} \geq p_{ji}^{(m)} \left(\sum_k p_{ii}^{(k)} \right) p_{ij}^{(n)} = \infty.$$

This shows that j is also recurrent. □

Remark. Similarly, if state i is transient and $i \longleftrightarrow j$, then j is also transient. This shows that in any communicating class, the states must be all recurrent, or all transient. This further implies that **the states of a finite, irreducible Markov chain must all be recurrent**.

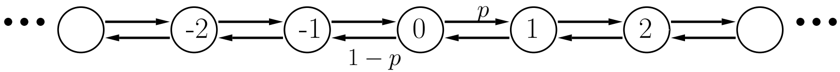
Example 0.4. Consider a Markov chain consisting of five states $\{1, 2, 3, 4, 5\}$ and having the following transition probability matrix

$$\mathbf{P} = \begin{pmatrix} .4 & .3 & .3 & & \\ .6 & & & .4 & \\ .5 & .5 & & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix}$$

Determine which states are recurrent and which ones are transient.

Example 0.5. Consider a random walk with state space $S = \mathbb{Z}$ and transition probabilities:

$$p_{i,i+1} = p = 1 - p_{i,i-1}, \quad i \in \mathbb{Z}.$$



Clearly, all states communicate with each other, so they must all be recurrent or transient. Which case is it?

Solution. Consider state 0 for which $p_{00}^{(n)} = 0$ for all odd $n \geq 1$. Then

$$\sum_{n=1}^{\infty} p_{00}^{(n)} = \sum_{k=1}^{\infty} p_{00}^{(2k)} = \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k = \sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2} (p(1-p))^k$$

Using Stirling's approximation

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

and simplifying things gives that

$$\sum_{n=1}^{\infty} p_{00}^{(n)} \sim \sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{\sqrt{\pi n}}$$

It diverges if $p = \frac{1}{2}$ and is finite if $p \neq \frac{1}{2}$. This shows that state 0 is recurrent if $p = \frac{1}{2}$, or transient if $p \neq \frac{1}{2}$.

Remark.

- In the two-dimensional symmetric random walk over $\mathbb{R}^2$, all states are recurrent as well;
- However, in 3D or higher dimensions, all states of the symmetric random walk will be transient.

Def 0.6. For a fixed state i, let

$$d = \gcd\left\{n \geq 1 \mid p_{ii}^{(n)} > 0\right\}.$$

We say that state i is

- periodic with period d, if $d > 1$ (that is, return to state i is possible only in multiples of d time steps).
- aperiodic, if $d = 1$.

If all states of a Markov chain are periodic with the same period d (or aperiodic), then the chain is said to be periodic with period d (or aperiodic).

Theorem 0.4. If two states of a Markov chain communicate, then they have the same period. Thus, **periodicity is a class property**.

Proof. Suppose state i has a period of d, and state j has a period of d'. We would like to show that $d = d'$.

Since $i \longleftrightarrow j$, there exist $m, n \geq 0$ such that

$$p_{ij}^{(m)} > 0, \quad p_{ji}^{(n)} > 0.$$

and thus,

$$p_{ii}^{(m+n)} \geq p_{ij}^{(m)} p_{ji}^{(n)} > 0$$

From this, we conclude that $d \mid m+n$.

Now suppose that $p_{jj}^{(k)} > 0$ for some k. Since

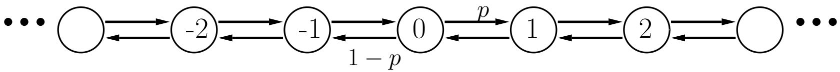
$$p_{ii}^{(m+n+k)} \geq p_{ij}^{(m)} p_{jj}^{(k)} p_{ji}^{(n)} > 0$$

we have $d \mid m+n+k$. We already know that $d \mid m+n$. It follows that $d \mid k$. This implies that $d \leq d'$.

By similar reasoning, we can show that $d' \mid d$.

Therefore, we must have $d = d'$. □

Example 0.6. Every state of the 1D random walk is periodic with period 2, since return to any starting point is only possible after an even number of steps.



Example 0.7. What is an example of a state that has a period of 3? (If there is a self-loop, then the period must be 1)

Let i be a recurrent state in a Markov chain. Define

$$N_i = \min\{n \geq 1 : X_n = i\} \quad \longleftarrow \text{ hitting time (first passage time)}$$

which counts the number of time steps needed for the chain to first enter state i (regardless of the initial state), and

$$m_{ii} = \mathrm{E}\left[N_i \mid X_0 = i\right] \quad \longleftarrow \text{ mean recurrence time}$$

which denotes the expected number of transitions that the chain takes to return to state i, given that it starts in state i.

Def 0.7. A recurrent state i is called positive recurrent if $m_{ii} < \infty$; it is called null recurrent if $m_{ii} = \infty$.

Remark. It can be shown that (we will prove this in next lecture)

- Positive recurrence is a class property, that is, if $i \longleftrightarrow j$ and i is positive recurrent, then j must be positive recurrent as well. Similarly, null recurrence is also a class property.

- In a finite Markov chain, all recurrent states are positive recurrent.

We derive a formula for the mean recurrence time m_{ii} at state i.

Let

$$f_{ii}^{(n)} = P(N_i = n \mid X_0 = i)$$

which represents the probability that we start from state i and return to it for the first time after n steps.

Then we have the following result.

Theorem 0.5.

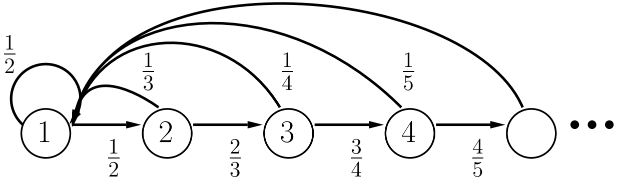
$$m_{ii} = \sum_{n=1}^{\infty} n \cdot f_{ii}^{(n)}.$$

Proof.

$$m_{ii} = \mathrm{E}(N_i \mid X_0 = i) = \sum_{n=1}^{\infty} nP(N_i = n \mid X_0 = i) = \sum_{n=1}^{\infty} n f_{ii}^{(n)}.$$

Example 0.8. Consider a Markov chain whose states are the positive integers and whose transition probabilities are

$$p_{i1} = \frac{1}{i+1}, \quad p_{i,i+1} = \frac{i}{i+1}, \qquad \text{for } i = 1, 2, \ldots$$



That is with increasingly large probability the chain will continue "stepping" to the right or alternatively "reset" to 1. Is state 1 recurrent? Is state 1 null recurrent?

Solution. State 1 is recurrent but not positive recurrent (thus null recurrent). First, to see that it is a recurrent state, compute

$$\begin{aligned} f_{11} &= 1 - P(\text{The chain never returns to } i \mid X_0 = i) \\ &= 1 - \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} \cdots = 1. \end{aligned}$$

Next, we compute its mean recurrence time to show it is null recurrent:

$$m_{11} = \sum_{n=1}^{\infty} n f_{11}^{(n)} = \sum_{n=1}^{\infty} n \cdot \frac{1}{2}\frac{2}{3} \cdots \frac{n-1}{n} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty. \quad \square$$

Recall that state i is transient if the probability that starting in state i, the process will ever reenter i satisfies

$$f_{ii} < 1.$$

Below is alternative way to check transiency.

Theorem 0.6. State i is transient if

$$\sum_{n=1}^{\infty} f_{ii}^{(n)} < 1.$$

Proof.

$$
\begin{aligned}
f_{ii} &= P\left(\bigcup_{n=1}^{\infty} \{X_n = i\} \middle| X_0 = i \right) \\
&= P(N_i < \infty \mid X_0 = i) \\
&= P\left(\bigcup_{n=1}^{\infty} \{N_i = n\} \middle| X_0 = i \right) \\
&= \sum_{n=1}^{\infty} P(N_i = n \mid X_0 = i) \\
&= \sum_{n=1}^{\infty} f_{ii}^{(n)}. \quad \square
\end{aligned}
$$

Def 0.8. A state which is positive recurrent and aperiodic is called **ergodic**. In other words, a state is ergodic if it

- is recurrent,
- has finite mean recurrence time, and
- has a period of 1.

If all states in an irreducible Markov chain are ergodic, then the chain is said to be ergodic.

Summary

Terminology

- n-step transition probability: $p_{ii}^{(n)}$
- Probability of ever re-entering state i (when starting from it): f_{ii}
- Hitting time (first passage time): N_i
- n-step hitting probability: $f_{ii}^{(n)}$
- Mean recurrence time: m_{ii}

Classification of states:

- Recurrent / transient
- Positive recurrent / null recurrent
- Periodic / aperiodic
- Ergodic