San José State University

Math 263: Stochastic Processes

Stationary distributions and limiting probabilities

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This lecture is based on the following textbook sections:

- Section 4.4

and also the following lecture: `https://www.stat.uchicago.edu/~yibi/teaching/stat317/2013/Lectures/Lecture5_4up.pdf`

Outline of the presentation

- Stationary distributions
- Limiting probabilities
- Long-run proportions

Assume a Markov chain $\{X_n : n = 0, 1, 2, \ldots\}$ with state space S and transition matrix $\mathbf{P}$.

Let $\boldsymbol{\pi} = (\pi_i)_{i \in S}$ be a row vector denoting a probability distribution on S, i.e.,

$$\pi_i \geq 0, \quad \sum_{i \in S} \pi_i = 1.$$

Def 0.1. $\boldsymbol{\pi}$ is called a **stationary** (or equilibrium) distribution of the Markov chain if it satisfies

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}, \qquad (\boldsymbol{\pi} \text{ is a left eigenvector corresponding to } 1)$$

or in entrywise form,

$$\pi_j = \sum_{i \in S} \pi_i p_{ij}, \quad \text{for all } j \in S.$$

Remark. $\boldsymbol{\pi}^T$ is a (right) eigenvector of $\mathbf{P}^T$ corresponding to the same eigenvalue 1:

$$\mathbf{P}^T \boldsymbol{\pi}^T = \boldsymbol{\pi}^T .$$

Note that $\mathbf{1}$ is a (right) eigenvector of $\mathbf{P}$ corresponding to eigenvalue 1:

$$\mathbf{P}\mathbf{1} = \mathbf{1} .$$

In general, a square matrix $\mathbf{A}$ and its transpose have the same eigenvalues

$$\det(\lambda \mathbf{I} - \mathbf{A}^T) = \det(\lambda \mathbf{I} - \mathbf{A})$$

but they do not have the same eigenvectors.

Theorem 0.1. Let $\{X_n : n = 0, 1, 2, \ldots\}$ be a Markov chain with a stationary distribution $\boldsymbol{\pi}$. If $X_n \sim \boldsymbol{\pi}$ for some integer $n \geq 0$, then $X_{n+1} \sim \boldsymbol{\pi}$.

Remark. This implies that for the same n, the future states X_{n+2}, X_{n+3}, … all have the same distribution $\boldsymbol{\pi}$.

Proof. For any $j \in S$,

$$\begin{aligned} P(X_{n+1} = j) &= \sum_{i \in S} P(X_{n+1} = j \mid X_n = i) P(X_n = i) \\ &= \sum_{i \in S} p_{ij} \pi_i = \pi_j. \qquad \square \end{aligned}$$

Example 0.1. Find the stationary distribution of the Markov chain below:

$$\mathbf{P} = \begin{pmatrix} 0 & .9 & .1 & 0 \\ .8 & .1 & 0 & .1 \\ 0 & .5 & .3 & .2 \\ .1 & 0 & 0 & .9 \end{pmatrix}$$

Answer: $\boldsymbol{\pi} = (.2788, .3009, .0398, .3805)$ by software. Alternatively, we can solve $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ (along with the requirement $\sum \pi_i = 1$) directly by hand:

$$\begin{cases} \pi_1 = & 0.8\pi_2 + 0.1\pi_4 \\ \pi_2 = & 0.9\pi_1 + 0.1\pi_2 + 0.5\pi_3 \\ \pi_3 = & 0.1\pi_1 + 0.3\pi_3 \\ \pi_4 = & 0.1\pi_2 + 0.2\pi_3 + 0.9\pi_4 \end{cases} \longrightarrow \begin{cases} \pi_1 = & 63/226 \\ \pi_2 = & 68/226 \\ \pi_3 = & 9/226 \\ \pi_4 = & 86/226 \end{cases}$$

Existence (and uniqueness) of stationary distributions

Theorem 0.2. For any irreducible Markov chain with state space S and transition matrix $\mathbf{P}$, it has a stationary distribution $\boldsymbol{\pi} = (\pi_j)$:

$$\forall j \in S: \ \pi_j \geq 0, \quad \sum_{i\in S} \pi_i = 1, \quad \boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}.$$

if and only if the chain is positive recurrent.

Furthermore, if a solution exists, then it will be unique and for state j,

$$\pi_j = \begin{cases} \lim_{n\to\infty} p_{ij}^{(n)}, & \text{if the chain is aperiodic} \\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)}, & \text{if the chain is periodic} \end{cases}$$

Remark. In the aperiodic case, π_j is also the limiting probability that the chain is in state j, i.e.,

$$\pi_j = \lim_{n\to\infty} P(X_n = j).$$

To prove this, let $\boldsymbol{\alpha} = (\alpha_i)_{i\in S}$ be the initial distribution of the chain.

Then

$$\begin{aligned} P(X_n = j) &= \sum_{i\in S} P(X_n = j \mid X_0 = i)\, P(X_0 = i) \\ &= \sum_{i\in S} p_{ij}^{(n)} \alpha_i \quad \xrightarrow{n\to\infty} \quad \pi_j \sum_{i\in S} \alpha_i = \pi_j. \end{aligned}$$

Example 0.2 (Social mobility)**.** Let X_n be a family's social class: 1 (lower), 2 (middle), 3 (upper) in the nth generation. This was modeled as a Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} .8 & .1 & .1 \\ .2 & .6 & .2 \\ .3 & .3 & .4 \end{pmatrix}$$

It is irreducible, positive recurrent and aperiodic (i.e., ergodic). Thus, there is a unique stationary distribution:

$$\boldsymbol{\pi} = \left(\frac{6}{11}, \frac{3}{11}, \frac{2}{11}\right) = (0.5454, 0.2727, 0.1818),$$

and the chain will converge to the stationary distribution.

$$\mathbf{P} = \begin{pmatrix} .8 & .1 & .1 \\ .2 & .6 & .2 \\ .3 & .3 & .4 \end{pmatrix} \longrightarrow \mathbf{P}^{10} = \begin{pmatrix} 0.5471 & 0.2715 & 0.1814 \\ 0.5430 & 0.2745 & 0.1825 \\ 0.5441 & 0.2737 & 0.1822 \end{pmatrix}$$

$$\longrightarrow \mathbf{P}^{20} = \begin{pmatrix} 0.5455 & 0.2727 & 0.1818 \\ 0.5454 & 0.2727 & 0.1818 \\ 0.5454 & 0.2727 & 0.1818 \end{pmatrix}$$

Example 0.3. Consider the following Markov chain:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is irreducible and positive recurrent, and thus has a unique stationary distribution:

$$\boldsymbol{\pi} = \left(\frac{1}{2}, \frac{1}{2}\right).$$

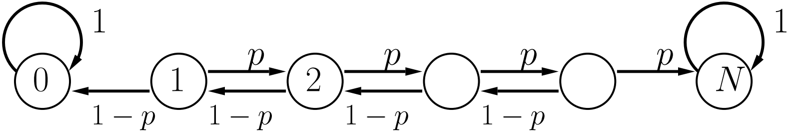
The chain does not converge to the stationary distribution because it is periodic with period 2: For any integer $\ell \geq 0$,

$$\mathbf{P}^{2\ell} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{P}^{2\ell+1} = \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

However, the following identity is still true:

$$\pi_j = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(k)}$$

Example 0.4 (Gambler's Ruin)**.** The underlying Markov chain has three communicating classes $\{0\}, \{1, \ldots, N-1\}, \{N\}$, and thus it is not irreducible.



However, the chain has two stationary distributions (corresponding to the two recurrent classes):

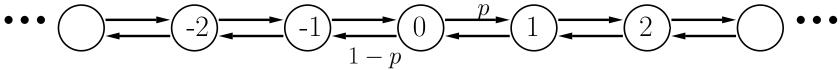
$$\boldsymbol{\pi}_1 = (1, 0, \ldots, 0, 0), \quad \boldsymbol{\pi}_2 = (0, 0, \ldots, 0, 1)$$

When $N = 4$ and $= \frac{1}{2}$ (symmetric random walk),

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \longrightarrow \quad \mathbf{P}^{30} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

What does this imply?

Example 0.5. The 1-dimensional symmetric random walk over $\mathbb{Z}$ must be null recurrent.

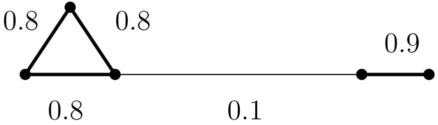


(This is a homework question, #39. Use proof by contradiction)

Consider the Markov chain defined on a finite, undirected, weighted graph $\mathscr{G} = \{V, E, \mathbf{W}\}$, with state space $S = V$ and transition matrix

$$\mathbf{P} = \mathbf{D}^{-1}\mathbf{W}, \qquad \mathbf{D} = \mathrm{diag}(\mathbf{d}), \quad \mathbf{d} = \mathbf{W} \cdot \mathbf{1}$$

The chain is finite, and if the graph is connected, then the Markov chain must be irreducible and also positive recurrent. Accordingly, it possesses a unique stationary distribution.



Proposition 0.3. For any finite, connected graph, the induced Markov chain possesses the following unique stationary distribution

$$\boldsymbol{\pi} = \frac{1}{\mathrm{Vol}(V)} \cdot \mathbf{d}, \qquad \text{where} \quad \mathrm{Vol}(V) = \sum_{i \in V} d_i.$$

If the graph is also non-bipartite, then the chain always converges to the above stationary distribution.

Proof. First, we show that

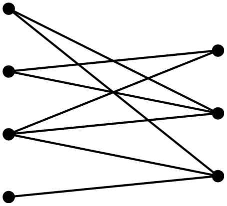
$$\mathbf{dP} = \mathbf{dD}^{-1}\mathbf{W} = \mathbf{1}^T\mathbf{W} = \mathbf{d} \qquad \longrightarrow \quad \boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}.$$

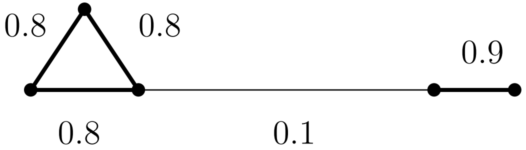
Thus, $\boldsymbol{\pi}$ is a stationary distribution of the chain and it is also unique.

For the convergence part, we consider the following two cases:

(1) Bipartite graphs (no convergence, because $d = 2$)

(2) Non-bipartite graphs (convergence)





Long-run proportion of visits to a state

Theorem 0.4. For an irreducible, positive recurrent Markov chain with stationary distribution $\boldsymbol{\pi} = (\pi_j)$, π_j is also the long-run proportion of time that the chain is in state j (regardless of initial state i).

$$\underset{X_1}{\_\_\_}\ \underset{X_2}{\_\_\_}\ \underline{\ j\ }\ \_\_\_\ \_\_\_\ \_\_\_\ \underline{\ j\ }\ \_\_\_\ \underline{\ j\ }\ \_\_\_\ \_\_\_\ \underline{\ j\ }\ \_\_\_\ \underset{X_\ell}{\_\_\_}$$

Proof. To see this, let

$$I_n = 1_{X_n=j}, \quad \text{for all } n \geq 1$$

and define

$$T = \sum_{n=1}^{\ell} I_n$$

which represents the total number of visits to state j in ℓ steps.

The proportion of visits to state j in ℓ steps is

$$\frac{T}{\ell} = \frac{1}{\ell}\sum_{n=1}^{\ell} I_n,$$

and we would like to show that it converges to π_j on average:

$$\begin{aligned}
\mathrm{E}\left[\frac{T}{\ell}\,\middle|\, X_0 = i\right] &= \frac{1}{\ell}\sum_{n=1}^{\ell} \mathrm{E}[I_n \mid X_0 = i] \\
&= \frac{1}{\ell}\sum_{n=1}^{\ell} 1\cdot P(I_n = 1 \mid X_0 = j) + 0\cdot P(I_n = 0 \mid X_0 = j) \\
&= \frac{1}{\ell}\sum_{n=1}^{\ell} P(X_n = j \mid X_0 = j) \\
&= \frac{1}{\ell}\sum_{n=1}^{\ell} p_{ij}^{(n)} \quad \xrightarrow{\ell\to\infty} \quad \pi_j.
\end{aligned}$$

Example 0.6. Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

$$C \quad C \quad T \quad C \quad C \quad C \quad T \quad T \quad C \quad C \quad C \quad C \quad T \quad C \quad C \quad C$$

Solution. Let X_n be the type of the nth vehicle, T (for truck) or C (for car), when counting from one end of the road to the other end. Then $\{X_n, n \geq 1\}$ is a Markov chain with state space $S = \{T, C\}$ and corresponding transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

Since the chain is irreducible and positive recurrent, it has a unique stationary distribution $\boldsymbol{\pi} = (\pi_T, \pi_C)$ given by

$$\pi_T = \pi_T \cdot \frac{1}{4} + \pi_C \cdot \frac{1}{5}, \quad \pi_T + \pi_C = 1 \qquad \longrightarrow \qquad \pi_T = \frac{4}{19},\ \pi_C = \frac{15}{19}$$

The fraction of trucks on the road is the long-run proportion $\pi_T = \frac{4}{19}$.

Theorem 0.5. For any irreducible, positive recurrent Markov chain, with stationary distribution $\boldsymbol{\pi} = (\pi_j)$, we must have

$$\pi_j = \frac{1}{m_{jj}} \quad \text{for all } j \in S,$$

where m_{jj} represents the mean recurrence time of state j:

$$m_{jj} = \mathrm{E}(N_j \mid X_0 = j).$$

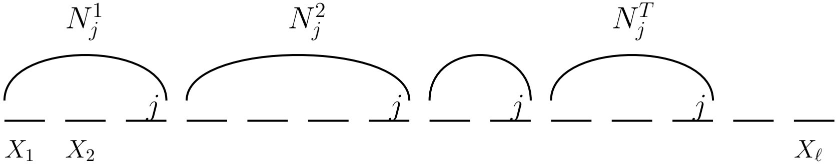
Remark. This theorem implies that $\pi_j > 0$ for all positive recurrent states j in an irreducible chain (as $m_{jj} < \infty$ for all j). Note that π_j can also be interpreted as the long-run proportion of the chain being in state j here.

Proof. To see this, consider

$$T = \sum_{n=1}^{\ell} I_n, \quad I_n = 1_{X_n = j}$$

which represents the total number of visits to state j in ℓ time steps.

Denote by $N_j^1, \ldots, N_j^T$ the individual recurrence times in the ℓ time steps:



Then

$$N_j^1 + \cdots + N_j^T \le \ell < N_j^1 + \cdots + N_j^T + N_j^{T+1},$$

where N_j^{T+1} represents the additional number of time steps that will be needed by the chain to enter state j again (after the first T visits).

Taking conditional expectation $\mathrm{E}[\cdot \mid X_0 = j]$ of left-hand side gives that

$$\begin{aligned}
\mathrm{E}\left(N_j^1 + \cdots + N_j^T \,\middle|\, X_0 = j \right) &= \mathrm{E}\left[\mathrm{E}\left(N_j^1 + \cdots + N_j^T \,\middle|\, X_0 = j, T \right) \middle| X_0 = j \right] \\
&= \mathrm{E}\left[T \cdot \mathrm{E}\left(N_j^1 \,\middle|\, X_0 = j \right) \middle| X_0 = j \right] \\
&= \mathrm{E}\left[T \cdot m_{jj} \,\middle|\, X_0 = j \right] \\
&= m_{jj} \cdot \mathrm{E}\left[T | X_0 = j \right]
\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{E}\left(N_j^1 + \cdots + N_j^T + N_j^{T+1} \,\middle|\, X_0 = j \right) &= m_{jj} \cdot \mathrm{E}\left[T + 1 | X_0 = j \right] \\ &= m_{jj} + m_{jj} \cdot \mathrm{E}\left[T | X_0 = j \right]\end{aligned}$$

Combining them together, we have

$$m_{jj} \cdot \mathrm{E}[T \mid X_0 = j] \le \ell < m_{jj} + m_{jj} \cdot \mathrm{E}[T \mid X_0 = j]$$

or

$$m_{jj} \cdot \frac{1}{\ell} \mathrm{E}[T \mid X_0 = j] \le 1 < m_{jj} \left(\frac{1}{\ell} + \frac{1}{\ell} \mathrm{E}[T \mid X_0 = j] \right)$$

We next derive an expression for $\mathrm{E}[T \mid X_0 = j]$:

$$\begin{aligned}
\mathrm{E}[T \mid X_0 = j] &= \sum_{n=1}^{\ell} \mathrm{E}(I_n \mid X_0 = j) \\
&= \sum_{n=1}^{\ell} 1 \cdot P(I_n = 1 \mid X_0 = j) + 0 \cdot P(I_n = 0 \mid X_0 = j) \\
&= \sum_{n=1}^{\ell} P(X_n = j \mid X_0 = j) \\
&= \sum_{n=1}^{\ell} p_{jj}^{(n)}
\end{aligned}$$

It follows that

$$m_{jj} \cdot \frac{1}{\ell} \sum_{n=1}^{\ell} p_{jj}^{(n)} \leq 1 < m_{jj} \cdot \left(\frac{1}{\ell} + \frac{1}{\ell} \sum_{n=1}^{\ell} p_{jj}^{(n)} \right)$$

Letting $\ell \to \infty$ yields that

$$m_{jj} \cdot \pi_j \leq 1 \leq m_{jj} \cdot (0 + \pi_j)$$

So we must have

$$m_{jj} \cdot \pi_j = 1, \qquad \text{and thus} \quad \pi_j = \frac{1}{m_{jj}}.$$

Remark. If the chain is irreducible but null recurrent, then $m_{jj} = \infty$ for all states j. Such a Markov chain may have no stationary distribution $\boldsymbol{\pi}$ (e.g., the 1D symmetric random walk over $\mathbb{Z}$).

However, we can still talk about the long-run proportion of the chain being in state j:

$$\frac{T}{\ell} = \frac{1}{\ell} \sum_{n=1}^{\ell} I_n, \quad \text{as } \ell \to \infty.$$

Starting with the inequality

$$N_j^1 + \cdots + N_j^T \le \ell \quad \text{for all } \ell$$

we take conditional expectation $\mathrm{E}[\cdot \mid X_0 = j]$ and repeat the same steps to obtain that

$$m_{jj} \cdot \mathrm{E}[T \mid X_0 = j] \le \ell \quad \text{for all } \ell$$

or equivalently,

$$m_{jj} \cdot \mathrm{E}[T/\ell \mid X_0 = j] \le 1 \quad \text{for all } \ell$$

Because state j is null recurrent $(m_{jj} = \infty)$, we must have

$$\mathrm{E}[T/\ell \mid X_0 = j] = 0 \quad \text{for all } \ell$$

This shows that the long run proportion of visits to state j is zero. Thus, if π_j represents the long-run proportion of state j (instead of a stationary probability), then the formula $\pi_j = \frac{1}{m_{jj}}$ is still valid.

Theorem 0.6. **Positive recurrence is a class property**. That is, if state j is positive recurrent, and state j communicates with state k, then state k is also positive recurrent.

Proof. (We cannot use the stationary distribution as we do not know whether it exists; we'll consider long-run proportions instead)

First, there exists a positive integer n such that

$$p_{jk}^{(n)} > 0$$

Since state j is positive recurrent, the long-run proportion is

$$\pi_j = 1/m_{jj} > 0$$

For any positive integer t and state i, we have

$$p_{ik}^{(t+n)} \geq p_{ij}^{(t)} \cdot p_{jk}^{(n)}$$

and also

$$\frac{1}{\ell}\sum_{t=1}^{\ell} p_{ik}^{(t+n)} \geq \left(\frac{1}{\ell}\sum_{t=1}^{\ell} p_{ij}^{(t)}\right) \cdot p_{jk}^{(n)}$$

Letting $\ell \to \infty$, we obtain that

$$\pi_k \geq \pi_j \cdot p_{jk}^{(n)} > 0$$

where π_k represents the long-run proportion of visits to state k. It follows that

$$m_{kk} = \frac{1}{\pi_k} < \infty$$

and thus state k is also positive recurrent. □