San José State University

Math 263: Stochastic Processes

Time-reversible Markov chains

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This lecture is based on the following textbook sections:

- Section 4.8

Outline of the presentation

- Time-reversibility

Assume an irreducible, ergodic Markov chain with transition matrix $\mathbf{P}$ and stationary distribution $\boldsymbol{\pi}$.

Suppose it has already been running for a long time.

Given a current time n (which is a large number), we trace the sequence of states going back in time:

$$\ldots,\ \underbrace{X_{n-2}}_{Y_2},\ \underbrace{X_{n-1}}_{Y_1},\ \underbrace{X_n}_{Y_0},\ \ldots$$

It turns out that $\{Y_k = X_{n-k} : k = 0, 1, 2, \ldots\}$ is also a Markov chain.

Theorem 0.1. The stochastic process $\{Y_k, k \geq 0\}$ is a Markov chain with transition probabilities

$$q_{ij} = \frac{\pi_j p_{ji}}{\pi_i}$$

Proof. We prove only that

$$P(X_m = j \mid X_{m+1} = i, X_{m+2} = k) = \underbrace{P(X_m = j \mid X_{m+1} = i)}_{q_{ij}}$$

By Bayes Rule,

$$
\begin{aligned}
&P(X_m = j \mid X_{m+1} = i, X_{m+2} = k) \\
&= \frac{P(X_{m+1} = i, X_{m+2} = k \mid X_m = j) P(X_m = j)}{P(X_{m+1} = i, X_{m+2} = k)} \\
&= \frac{P(X_{m+2} = k \mid X_{m+1} = i, \cancel{X_m = j}) P(X_{m+1} = i \mid X_m = j) P(X_m = j)}{P(X_{m+2} = k \mid X_{m+1} = i) P(X_{m+1} = i)} \\
&= \frac{P(X_{m+1} = i \mid X_m = j) P(X_m = j)}{P(X_{m+1} = i)} \qquad \longleftarrow P(X_m = j \mid X_{m+1} = i) \\
&= \frac{p_{ji} \pi_j}{\pi_i} \qquad \longleftarrow q_{ij}
\end{aligned}
$$

Def 0.1. A stationary, ergodic Markov chain $\{X_n : n = 0, 1, 2, \ldots\}$ is said to be time-reversible if its transition probabilities are the same as those of the reversed Markov chain, i.e.,

$$p_{ij} = q_{ij}, \qquad \text{for all } i, j \in S$$

Remark. The condition of the definition is equivalent to

$$\pi_i p_{ij} = \pi_j p_{ji}, \qquad \text{for all } i, j \in S$$

i.e.,

$$P(X_n = i, X_{n+1} = j) = P(X_n = j, X_{n+1} = i)$$

That is, as the chain has converged to its stationary distribution, it is equally often to transition from i to j and from j to i.

Remark. If there exist two states such that $p_{ij} > 0$ but $p_{ji} = 0$, then the chain is not reversible.

Example 0.1. Show that any Markov chain induced from an undirected, weighted graph $\mathscr{G} = (V, E, \mathbf{W})$ is time-reversible.

Proof.

$$\pi_i p_{ij} = \frac{d_i}{\mathrm{Vol}(V)} \frac{w_{ij}}{d_i} = \frac{w_{ij}}{\mathrm{Vol}(V)} = \frac{w_{ji}}{\mathrm{Vol}(V)} = \pi_j p_{ji}.$$

The following theorem indicates that time reversibility implies existence of a stationary distribution.

Theorem 0.2. Assume an irreducible, ergodic Markov chain. If there exists a distribution $\mathbf{x} = (x_i)$ such that

$$x_i p_{ij} = x_j p_{ji}, \quad \forall\, i, j$$

then

- $\mathbf{x}$ is the stationary distribution of the chain, and
- the chain is time reversible.

Proof.

$$\sum_j x_j p_{ji} = \sum_j x_i p_{ij} = x_i \sum_j p_{ij} = x_i.$$

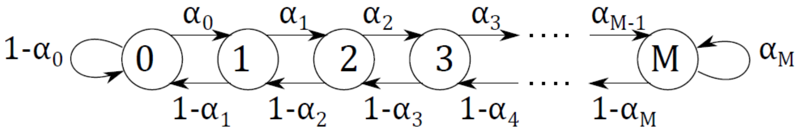
In vector form, this is

$$\mathbf{x}\mathbf{P} = \mathbf{x}.$$

Since the chain is irreducible and positive recurrent, $\mathbf{x}$ must be its unique stationary distribution.

Example 0.2. Determine if the following Markov chain is time-reversible:

$$p_{i,i+1} = \alpha_i = 1 - p_{i,i-1}, \quad i = 1, \ldots, M-1$$
$$p_{0,1} = \alpha_0 = 1 - p_{0,0}$$
$$p_{M,M} = \alpha_M = 1 - p_{M,M-1}$$



Solution. The chain is time-reversible, which can be formally justified by solving the equations $\pi_i p_{ij} = \pi_j p_{ji}$ for all i, j:

$$\begin{aligned} \pi_0 \alpha_0 &= \pi_1 (1-\alpha_1) \\ \pi_1 \alpha_1 &= \pi_2 (1-\alpha_2) \\ &\vdots \\ \pi_{M-1} \alpha_{M-1} &= \pi_M (1-\alpha_M) \end{aligned}$$

It is straightforward to show that

$$\pi_0 = \left[1 + \sum_{i=1}^{M} \frac{\alpha_0 \cdots \alpha_{i-1}}{(1-\alpha_1)\cdots(1-\alpha_i)}\right]^{-1}, \quad \pi_j = \frac{\alpha_0 \cdots \alpha_{j-1}}{(1-\alpha_1)\cdots(1-\alpha_j)} \pi_0, \ 1 \le j \le M$$

Example 0.3 (Ehrenfest Model for Diffusion)**.** Suppose that M molecules are distributed among two urns; and at each time point one of the molecules is chosen at random, removed from its urn, and placed in the other one. The number of molecules in urn 1 is a special case of the Markov chain of the preceding example with

$$\alpha_i = \frac{M-i}{M}, \quad i = 0, 1, \ldots, M$$

It can be shown that

$$\pi_i = \frac{\binom{M}{i}}{2^M}, \quad i = 0, 1, \ldots, M$$