San José State University

Math 263: Stochastic Processes

Mean time spent in transient states

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This lecture is based on the following textbook sections:

- Section 4.6
- Section 4.5.1

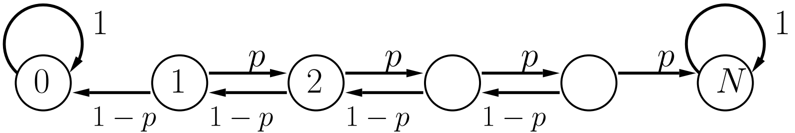
Outline of the presentation

- Mean time spend in transient states
- Transition probability between transient states
- The Gambler's Ruin problem

Mean time spent in transient states

Consider a finite-state Markov chain with transient states numbered as $\mathscr{T} = \{1, \ldots, t\}$ (and recurrent states numbered above t or under 1).

For example, in the Gambler's Ruin problem, let X_n denote the gambler's fortune after the nth bet. Then $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain:



The transient states are $1, \ldots, t = N - 1$ (and the recurrent states are $0, N$).

For any two transient states $i, j \in \mathscr{T}$, let s_{ij} denote the expected number of time periods that the Markov chain is in state j, given that it starts in state i:

$$s_{ij} = \mathrm{E}(T_0 \mid X_0 = i),$$

where

$$T_0 = \sum_{n=0}^{\infty} I_n, \quad I_n = 1_{X_n = j}$$

The following theorem shows how to compute all the s_{ij} collectively.

Theorem 0.1. Let $\mathbf{P}_{\mathscr{T}} = (p_{ij})_{1\le i,j\le t} \in \mathbb{R}^{t\times t}$, the transition matrix restricted to the transient states, and $\mathbf{S} = (s_{ij}) \in \mathbb{R}^{t\times t}$, the matrix of mean times in transient states (when starting in transient states). Then

$$\mathbf{S} = (\mathbf{I} - \mathbf{P}_{\mathscr{T}})^{-1}.$$

Proof. We condition on the initial transition:

$$\begin{aligned} s_{ij} &= \mathrm{E}(T_0 \mid X_0 = i) \\ &= \sum_k \mathrm{E}(T_0 \mid X_0 = i, X_1 = k) P(X_1 = k \mid X_0 = i) \\ &= \sum_k (\delta_{ij} + s_{kj}) p_{ik} = \delta_{ij} + \sum_k p_{ik} s_{kj} \\ &= \delta_{ij} + \sum_{k=1}^{t} p_{ik} s_{kj} \qquad (s_{kj} = 0 \text{ for recurrent states } k) \end{aligned}$$

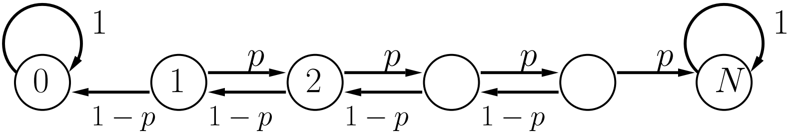
In matrix notation, this equation is

$$\mathbf{S} = \mathbf{I} + \mathbf{P}_{\mathscr{T}} \mathbf{S} \qquad \longrightarrow \qquad (\mathbf{I} - \mathbf{P}_T)\mathbf{S} = \mathbf{I}$$

From this we obtain that $\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1}$. □

Example 0.1 (Gamber's Ruin). Suppose $p = \frac{1}{2}$, $N = 5$. Then the transient states are $\mathscr{T} = \{1, 2, 3, 4\}$, and

$$\mathbf{P}_{\mathscr{T}} = \begin{pmatrix} & 0.5 & & \\ 0.5 & & 0.5 & \\ & 0.5 & & 0.5 \\ & & 0.5 & \end{pmatrix} \quad \longrightarrow \quad \mathbf{S} = \begin{pmatrix} 1.6 & 1.2 & 0.8 & 0.4 \\ 1.2 & 2.4 & 1.6 & 0.8 \\ 0.8 & 1.6 & 2.4 & 1.2 \\ 0.4 & 0.8 & 1.2 & 1.6 \end{pmatrix}$$



Transition probability between two states

Def 0.1. For any two states i, j, define by f_{ij} the probability that starting in state i, the process will ever make a transition into state j:

$$f_{ij} = P\left(\cup_{n=1}^{\infty} \{X_n = j\} \mid X_0 = i\right)$$

Remark. Compare with f_{ii} and $f_{ii}^{(n)}$.

Theorem 0.2. For any two transient states i, j,

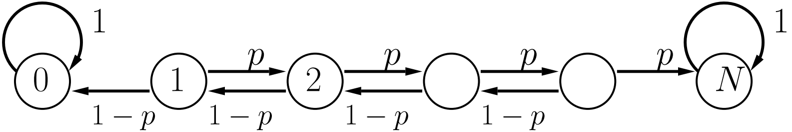
$$f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}$$

Proof. It follows from the following equation:

$$s_{ij} = \delta_{ij} + f_{ij} \cdot s_{jj} + \left(1 - f_{ij}\right) \cdot 0$$

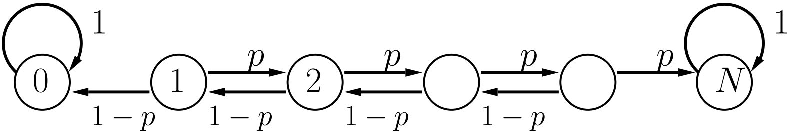
Example 0.2 (Cont'd). Starting the dollar amounts $1, 2, 3, 4$, the probabilities of the gambler ever reaching each of those amounts (again) are given by

$$\mathbf{S} = \begin{pmatrix} 1.6 & 1.2 & 0.8 & 0.4 \\ 1.2 & 2.4 & 1.6 & 0.8 \\ 0.8 & 1.6 & 2.4 & 1.2 \\ 0.4 & 0.8 & 1.2 & 1.6 \end{pmatrix} \quad \longrightarrow \quad \mathbf{F}_{\mathscr{T}} = \begin{pmatrix} 0.3750 & 0.5000 & 0.3333 & 0.2500 \\ 0.7500 & 0.5833 & 0.6667 & 0.5000 \\ 0.5000 & 0.6667 & 0.5833 & 0.7500 \\ 0.2500 & 0.3333 & 0.5000 & 0.3750 \end{pmatrix}$$



From transient to recurrent

Example 0.3. Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0?



Solution. Let $p_i = f_{iN}$ for $i = 1, \ldots, N-1$ and $q = 1 - p$ for convenience.

By conditioning on X_1 we get that

$$p_i = p \cdot p_{i+1} + q \cdot p_{i-1}, \quad i = 1, \ldots, N-1$$

where we have defined $p_0 = 0,\ p_N = 1$.

Write $p_i = (p+q) \cdot p_i$ and substitute it into the above recursive relation to get that

$$q(p_i - p_{i-1}) = p(p_{i+1} - p_i) \quad \longrightarrow \quad \frac{p_{i+1} - p_i}{p_i - p_{i-1}} = \frac{q}{p}$$

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It follows that

$$p_i - p_{i-1} = (p_1 - \cancel{p_0}) \left(\frac{q}{p}\right)^{i-1}, \quad i = 1, \ldots, N$$

and by telescoping,

$$\begin{aligned} p_i &= (p_i - p_{i-1}) + (p_{i-1} - p_{i-2}) + \cdots + (p_2 - p_1) + (p_1 - p_0) \\ &= p_1 \left(\frac{q}{p}\right)^{i-1} + p_1 \left(\frac{q}{p}\right)^{i-2} + \cdots + p_1 \left(\frac{q}{p}\right) + p_1 \\ &= \begin{cases} p_1 \cdot \frac{1-(q/p)^i}{1-(q/p)}, & p \neq q \\ p_1 \cdot i, & p = q \end{cases} \end{aligned}$$

To determine p_1, use $p_N = 1$:

$$p_1 = \begin{cases} \frac{1-(q/p)}{1-(q/p)^N}, & p \neq q \\ \frac{1}{N}, & p = q \end{cases}$$

Consequently,

$$p_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N}, & p \neq q \\ \frac{i}{N}, & p = q \end{cases}$$

Example 0.4 (Gambler's Ruin, cont'd)**.** , Assume the same setting as before. If the player quits gambling once he either reaches a fortune of N or goes broke (whatever comes first), how long on average will that take?

Solution. Let

$$T_i = \min\{n \ge 0 : X_n = 0 \text{ or } X_n = N \mid X_0 = i\}, \quad i = 1, \ldots, N-1$$

We would like to find $m_i = \mathrm{E}(T_i)$.

By conditioning on X_1 we get that

$$
\begin{aligned}
m_i &= \mathrm{E}(T_i \mid X_1 = i+1) P(X_1 = i+1 \mid X_0 = i) \\
&\quad + \mathrm{E}(T_i \mid X_1 = i-1) P(X_1 = i-1 \mid X_0 = i) \\
&= (1 + \mathrm{E}(T_{i+1})) \cdot p + (1 + \mathrm{E}(T_{i-1})) \cdot q \\
&= 1 + p \cdot m_{i+1} + q \cdot m_{i-1}, \quad i = 1, \ldots, N-1
\end{aligned}
$$

We solve the above recursive relations, along with boundary conditions $T_0 = T_N = 0$, only for the case of $p = q = 1/2$:

$$
m_i = i \cdot (N - i), \quad i = 0, 1, \ldots, N
$$

Remark. When $p \neq q$, it can be shown that

$$m_i = \frac{N}{p-q} \cdot \left[\frac{1-(q/p)^i}{1-(q/p)^N} - \frac{i}{N} \right], \quad i = 0, 1, \ldots, N$$