San José State University

Math 263: Stochastic Processes

Poisson processes

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This lecture is based on the following textbook sections:

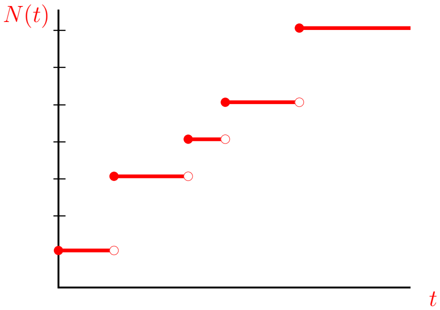
- Section 5.3 (5.3.1 - 5.3.4)
- Section 5.4 (5.4.1 - 5.4.2)

Outline of the presentation

- Counting processes
- Poisson processes
- Generalizations of Poisson processes

HW6: To be assigned in Canvas

Def 0.1. A stochastic process $\{N(t), t \geq 0\}$ is called a **counting process** if $N(t)$ represents the total number of events that occur by time t.



Remark. Any counting process $N(t)$ must satisfy:

- $N(t) \geq 0$;
- $N(t)$ is integer valued;
- If $s < t$, then $N(s) \leq N(t)$;
- For any $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

Def 0.2. Let $\{N(t), t \geq 0\}$ be a counting process.

- It is said to have **independent increments**, if the numbers of events that occur in disjoint time intervals are independent;

- It is said to have **stationary increments**, if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval.

Below is the definition of the a Poisson process (An equivalent alternative definition using $o(h)$ is given in the book).

Def 0.3. The counting process $\{N(t), t \geq 0\}$ is called a **Poisson process** with rate λ, if

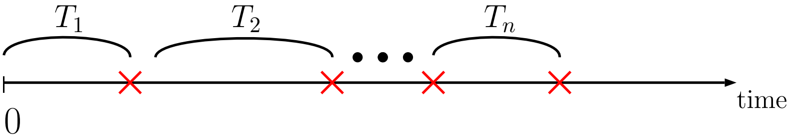
- $N(0) = 0$;
- The process has independent (and stationary) increments;
- The number of events in any interval of length t is Poisson distributed with mean λt. That is, for any $s, t \geq 0$:

$$P(N(t+s) - N(s) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

Consider a Poisson process:

- Denote the time of the first event by T_1.
- For any $n > 1$, let T_n denote the elapsed time between the $(n-1)$st and the nth event.

The sequence $\{T_n, n = 1, 2, \ldots\}$ is called the sequence of **interarrival times**.



Theorem 0.1. $\{T_n, n = 1, 2, \ldots\}$ are independent identically distributed exponential random variables with parameter λ.

Proof. The assumption of stationary and independent increments is basically equivalent to asserting that, at any point in time, the process probabilistically restarts itself.

Therefore, T_n are independently and identically distributed, and it is enough to determine the distribution of T_1:

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t} \qquad \longrightarrow \qquad T_1 \sim \text{Exp}(\lambda).$$

The following result then follows immediately (we proved this result at the beginning of the semester).

Corollary 0.2. The total waiting time for n occurrences of the event has a Gamma distribution (with parameters n, λ), i.e.,

$$S_n = T_1 + \cdots + T_n \sim \mathrm{Gamma}(n, \lambda)$$

This implies that

$$\mathrm{E}(S_n) = \frac{n}{\lambda}, \quad \mathrm{Var}(S_n) = \frac{n}{\lambda^2}.$$

Example 0.1. Suppose that people immigrate into a territory at a Poisson rate $\lambda = 10$ per week.

(a) What is the expected time until the 100th immigrant arrives?

(b) What is the probability that the elapsed time between the 100th and the 101st arrival exceeds one day?

It is also possible to define a Poisson process from a sequence of iid exponential random variables $\{T_n, n = 1, 2, \ldots\}$ with rate λ.

Theorem 0.3. Let

$$N(t) = \max\{n \geq 0 : T_1 + \cdots + T_n \leq t\}$$

Then $\{N(t), t \geq 0\}$ is a Poisson process with rate λ.

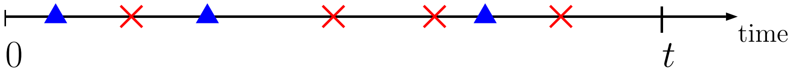
Proof. Fix an integer $n \geq 0$. Then $S_n = T_1 + \cdots + T_n \sim \text{Gamma}(n, \lambda)$ and it is independent of T_{n+1}.

By definition of $N(t)$,

$$\begin{aligned}
P(N(t)=n) &= P(S_n \le t, S_n + T_{n+1} > t) \\
&= \int_0^t \int_{t-s}^\infty f_{S_n}(s) f_{T_{n+1}}(x)\, \mathrm{d}x\, \mathrm{d}s \\
&= \int_0^t P(T_{n+1} > t-s) f_{S_n}(s)\, \mathrm{d}s \\
&= \int_0^t e^{-\lambda(t-s)} \frac{\lambda(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!}\, \mathrm{d}s \\
&= \frac{(\lambda t)^n e^{-\lambda t}}{n!}.
\end{aligned}$$

This shows that $N(t) \sim \mathrm{Pois}(\lambda t)$.

Consider a Poisson process $\{N(t), t \geq 0\}$ with rate λ, and suppose that each time the event occurs, it is classified as either a type I or a type II event, which occurs with probability p or $1-p$ respectively, independently of all other events.



Let $N_1(t)$ and $N_2(t)$ denote respectively the number of type I and type II events occurring in $[0,t]$. Note that $N(t) = N_1(t) + N_2(t)$.

Theorem 0.4. $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are both Poisson processes having respective rates λp and $\lambda(1-p)$. Furthermore, the two processes are independent.

Proof. For fixed $t > 0$,

$$
\begin{aligned}
P(N_1(t) = k) &= \sum_{n=k}^{\infty} P(N_1(t) = k \mid N(t) = n) P(N(t) = n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \sum_{n=k}^{\infty} \frac{p^k (1-p)^{n-k} (\lambda t)^n}{k!(n-k)!} e^{-\lambda t} \\
&= \frac{p^k (\lambda t)^k}{k!} e^{-(\lambda p)t} \sum_{m=0}^{\infty} \frac{(1-p)^m (\lambda t)^m}{m!} e^{-\lambda(1-p)t} \\
&= \frac{((\lambda p)t)^k}{k!} e^{-(\lambda p)t}, \quad k = 0, 1, 2, \ldots
\end{aligned}
$$

This shows that $N_1(t) \sim \text{Pois}(\lambda p)$ and similarly, $N_2(t) \sim \text{Pois}(\lambda(1-p))$.

To prove that the two processes are independent, consider for any $k, j \geq 0$:

$$\begin{aligned}
P(N_1(t) = k, N_2(t) = j) &= P(N_1(t) = k, N(t) = k + j) \\
&= P(N_1(t) = k \mid N(t) = k + j) \cdot P(N(t) = k + j) \\
&= \binom{k+j}{k} p^k (1-p)^j \cdot \frac{(\lambda t)^{k+j}}{(k+j)!} e^{-\lambda t} \\
&= \frac{p^k (1-p)^j (\lambda t)^{k+j}}{k! j!} e^{-\lambda t} \\
&= \frac{(\lambda p t)^k}{k!} e^{-\lambda p t} \cdot \frac{(\lambda (1-p) t)^j}{j!} e^{-\lambda (1-p) t} \\
&= P(N_1(t) = k) \cdot P(N_2(t) = j).
\end{aligned}$$

Example 0.2 (Cont'd)**.** If each immigrant is of certain descent with probability $\frac{1}{5}$, then what is the probability that no people of that descent will emigrate to the territory during the next two weeks?

Answer. $P(N_1(t) = 0) = e^{-2\cdot(10\cdot\frac{1}{5})} = .0183$

Example 0.3 (The Coupon Collecting Problem)**.** There are m different types of coupons. Each time a person collects a coupon (independently of ones previously obtained), it is a type j coupon with probability p_j ($p_j > 0$, $\sum p_j = 1$). Let N denote the number of coupons one needs to collect in order to have a complete collection of at least one of each type. Find $\mathrm{E}[N]$.

Solution. Suppose that coupons are collected at times chosen according to a Poisson process with rate $\lambda = 1$. Let $N_j(t)$ denote the number of type j coupons collected by time t. Then $\{N_j(t), t \geq 0\}, j = 1, \ldots, m$ are independent Poisson processes with respective rates $\lambda p_j = p_j$.

Let X_j denote the time of the first event of the jth process. Then

$$X = \max_{1 \leq j \leq m} X_j$$

is the time at which a complete collection is obtained.

Since the X_j are independent exponential random variables with respective rates p_j, it follows that

$$P(X < t) = P(X_1 < t, \ldots, X_m < t) = \prod_{j=1}^{m} (1 - e^{-p_j t}).$$

Therefore,

$$\mathrm{E}(X) = \int_0^\infty P(X > t)\,\mathrm{d}t = \int_0^\infty 1 - \prod_{j=1}^{m} (1 - e^{-p_j t})\,\mathrm{d}t.$$

It remains to relate it to $\mathrm{E}(N)$, the expected number of coupons it takes. To compute it, let T_i denote ith interarrival time of the Poisson process $N(t) = N_1(t) + \cdots + N_m(t)$. It is easy to see that

$$X = \sum_{i=1}^{N} T_i$$

from which we obtain that

$$\mathrm{E}(X) = \mathrm{E}(N) \cdot \mathrm{E}(T_1) = \mathrm{E}(N).$$

Therefore,

$$\mathrm{E}(N) = \int_0^\infty 1 - \prod_{j=1}^{m} (1 - e^{-p_j t})\, \mathrm{d}t.$$

Example 0.4 (The Coupon Collecting Problem, cont'd)**.** What is the expected number of coupon types that appear only once in the complete collection?

Solution. Let I_i be the indicator variable on whether there is only a single type i coupon in the final set, and $N_1 = \sum_{i=1}^{m} I_i$. Then

$$\mathrm{E}(N_1) = \sum_{i=1}^{m} \mathrm{E}(I_i) = \sum_{i=1}^{m} P(I_i = 1).$$

Note that there will be a single type i coupon in the final set if any other coupon type has appeared before the second coupon of type i is obtained.

Let $S_i \sim \text{Gamma}(2, p_i)$ denote the time at which the second type i coupon is obtained. Then

$$
\begin{aligned}
P(I_i = 1) &= P\left(\bigcap_{j \neq i} \{X_j < S_i\}\right) \\
&= \int_0^\infty P\left(\bigcap_{j \neq i} \{X_j < S_i\} \mid S_i = x\right) p_i^2 x e^{-p_i x} \, \mathrm{d}x \\
&= \int_0^\infty P\left(\bigcap_{j \neq i} \{X_j < x\} \mid \cancel{S_i = x}\right) p_i^2 x e^{-p_i x} \, \mathrm{d}x \\
&= \int_0^\infty \prod_{j \neq i} \left(1 - e^{-p_j x}\right) p_i^2 x e^{-p_i x} \, \mathrm{d}x
\end{aligned}
$$

It follows that

$$\mathrm{E}(N_1) = \int_0^\infty \sum_{i=1}^m \prod_{j\neq i} \left(1 - e^{-p_j x}\right) p_i^2 x e^{-p_i x}\,\mathrm{d}x$$

Remark. In the coupon collector problem when $m = 2$:

$$\mathrm{E}(N) = \frac{1}{p_1 p_2} - 1$$

$$\mathrm{E}(N_1) = 2 - p_1^2 - p_2^2$$

The next probability calculation related to Poisson processes is the probability that n events occur in one Poisson process before m events have occurred in a second and independent Poisson process.

More formally, let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent Poisson processes having respective rates λ_1 and λ_2.

Also, let $S_n^{(1)}$ denote the time of the nth event of the first process, and $S_m^{(2)}$ the time of the mth event of the second process.

Theorem 0.5.

$$P\left(S_n^{(1)} < S_m^{(2)}\right) = \sum_{k=n}^{n+m-1} \binom{k-1}{n-1} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^n \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{k-n}$$

Proof. In the special case of $n = m = 1$, where $S_1^{(i)} \sim \text{Exp}(\lambda_i), i = 1,2$ are independent, the formula reduces to

$$P\left(S_1^{(1)} < S_1^{(2)}\right) = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

This has been proved at the beginning of the semester.

To prove the general result, observe that each event that occurs is going to be

- an event of the $N_1(t)$ process with probability $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$, or
- an event of the $N_2(t)$ process with probability $1 - p = \frac{\lambda_2}{\lambda_1+\lambda_2}$,

independently of all that have previously occurred.

This question is thus equivalent to getting n heads before m tails when repeatedly flipping a coin with probability of heads p. $\square$

Suppose we are told that exactly one event of a Poisson process has taken place by time t, and we are asked to determine the distribution of the time at which the event occurred.

Theorem 0.6.

$$T_1 \mid N(t) = 1 \quad \sim \quad \mathrm{Unif}(0, t).$$

Proof. For $s < t$,

$$\begin{aligned}
P(T_1 < s \mid N(t) = 1) &= \frac{P(T_1 < s, N(t) = 1)}{P(N(t) = 1)} \\
&= \frac{P(N(s) = 1, N(t) - N(s) = 0)}{P(N(t) = 1)} \\
&= \frac{P(N(s) = 1)P(N(t) - N(s) = 0)}{P(N(t) = 1)} \\
&= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\
&= \frac{s}{t}.
\end{aligned}$$

Another interesting result is the joint distribution of the cumulative arrival times $S_1 \le \cdots \le S_n$ when given $N(t) = n$.

Theorem 0.7. Given that $N(t) = n$, $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent uniformly distributed random variables on $(0, t)$:

$$f(s_1, \ldots, s_n \mid N(t) = n) = \frac{n!}{t^n}, \quad 0 < s_1 < \cdots < s_n < t$$

(The proof of the theorem as well as its application is in Section 5.3.4)

Def 0.4. A stochastic process $\{X(t), t \geq 0\}$ is said to be a compound Poisson process if it has the form

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

where

- $\{N(t), t \geq 0\}$ is a Poisson process (with rate λ), and
- $\{Y_i\}$ are iid random variables that are independent of $N(t)$.

Theorem 0.8. In a compound Poisson process,

$$\mathrm{E}(X(t)) = \lambda t \mathrm{E}(Y_1), \quad \mathrm{Var}(X(t)) = \lambda t \mathrm{E}(Y_1^2)$$

Proof. Let μ, σ^2 be the expectation and variance of each Y_i. By direct calculation:

$$\begin{aligned} \mathrm{E}(X(t)) &= \mathrm{E}(N(t))\mu = \lambda t \mathrm{E}(Y_1), \\ \mathrm{Var}(X(t)) &= \mu^2 \mathrm{Var}(N(t)) + \sigma^2 \mathrm{E}(N(t)) \\ &= \lambda t(\mu^2 + \sigma^2) \\ &= \lambda t \mathrm{E}(Y_1^2). \end{aligned}$$

□

Example 0.5. Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. If the number of people in each family is independent and takes on the values $1, 2, 3, 4, 5$ with respective probabilities $1/4, 1/4, 1/3, 1/12, 1/12$, then what is the expected value and variance of the number of individuals migrating to this area during a fixed six-week period?

Answer. Let $N(t)$ be the number of families that migrate to the area over t weeks, and Y_i the size of each family. Then the number of individuals migrating to this area over t weeks is $X(t) = \sum_{i=1}^{N(t)} Y_i$.

Since

$$\mathrm{E}(Y_1) = \frac{5}{2} \qquad \text{and} \quad \mathrm{E}(Y_1^2) = \frac{1}{4} + 1 + 3 + \frac{41}{12} = \frac{23}{3},$$

we have

$$\mathrm{E}(X(6)) = 2 \cdot 6 \cdot \frac{5}{2} = 30, \quad \mathrm{Var}(X(6)) = 2 \cdot 6 \cdot \frac{23}{3} = 92$$

Def 0.5. The counting process $\{N(t), t \geq 0\}$ is called a nonhomogeneous Poisson process with intensity function $\lambda(t), t \geq 0$, if

- $N(0) = 0$
- The process has independent increments
- For any $s, t \geq 0$:

$$P(N(s+t) - N(s) = n) = e^{-R(s,t)} R(s,t)^n / n!, \quad n = 0, 1, 2, \ldots$$

where

$$R(s,t) = \int_s^{s+t} \lambda(y)\, \mathrm{d}y \qquad (= \lambda t \text{ for constant function } \lambda(y) = \lambda).$$