Worksheet 7: Vector spaces

Example 0.46. The following are all vector spaces:

- The set of all functions $f : \mathbb{R} \mapsto \mathbb{R}$;
- The set of all infinite sequences $(a_1, a_2, \ldots, a_n, \ldots)$;
- The set of all matrices of a fixed size $\mathbf{A} \in \mathbb{R}^{m \times n}$.

Example 0.47. Consider the vector space $V = \mathbb{R}^2$.

- Any line going through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$. In contrast, any line not passing through the origin is NOT a subspace.
- In fact, the single-element subset containing only the origin $\{\mathbf{0}\}$ is also a subspace of $\mathbb{R}^2$. It is called the zero subspace.
- The full vector space $\mathbb{R}^2$ is also a subspace of itself (though also a trivial one).

Example 0.48. For the vector space $V = \mathbb{R}^3$,

- Lines and planes passing through the origin are proper subspaces.
- $\{\mathbf{o}\}$ and $\mathbb{R}^3$ are trivial subspaces.

Example 0.49. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

Example 0.50. Let V be the vector space of all functions $f : \mathbb{R} \mapsto \mathbb{R}$. Then $H = \{\text{All polynomial functions}\}$ is a subspace.

Example 0.51. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 10 \\ 3 & 6 & 9 & 10 \end{bmatrix}$$

Do the following:

- Determine if $\mathbf{b} = [1 \ -1 \ 1]^T$ lies in the column space of $\mathbf{A}$
- Find $\mathrm{Col}(\mathbf{A})$. Is $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ onto?

Example 0.52. Consider the same matrix $\mathbf{A}$ above.

- Determine if $\mathbf{x} = [1 \ -2 \ 1]^T$ and $\mathbf{y} = [-5 \ 0 \ 5 \ -3]^T$ lie in the column space of $\mathbf{A}$
- Find $\mathrm{Nul}(\mathbf{A})$. Is $T(\mathbf{x}) = \mathbf{Ax}$ one to one?

Example 0.53. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -5 & 7 \\ 3 & 7 & -8 \end{bmatrix}$$

Find its null and column spaces. Of which Euclidean spaces are they each a subspace?