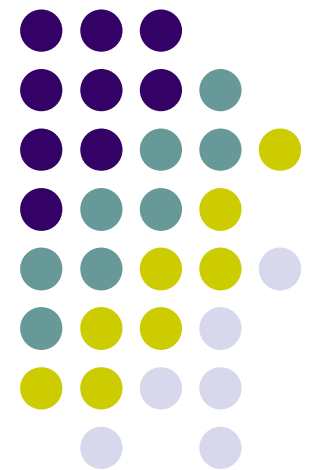
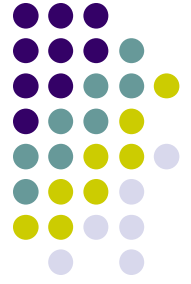


Advanced Vector Calculus

(optional)

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sjsu





Scalar (Dot) Product

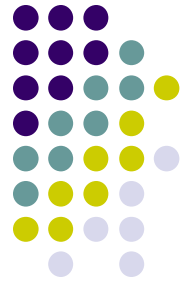
$$\vec{A} = \sum_{i=1}^3 A_i \hat{e}_i \equiv A_i \hat{e}_i$$

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \quad \text{Kronecker Delta Function} \quad \text{(orthogonal)}$$

$= 1$ (if $i = j$)
 $= 0$ (if $i \neq j$)

$$\vec{A} \cdot \vec{B} = A_i \hat{e}_i \cdot B_j \hat{e}_j = \delta_{ij} A_i B_j$$

$$\vec{A} \cdot \vec{B} = A_i B_i$$



Permutation Symbol ϵ_{ijk}

ϵ_{ijk}

= 1 (if ijk are cyclic permutation of 123)

= -1 (if ijk are non-cyclic permutation of 123; interchange a pair)

= 0 (if i, j or k is a duplicated index)

e.g. $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

$$\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$$

$$\epsilon_{113} = \epsilon_{221} = \epsilon_{322} = 0 \dots \text{etc}$$



Vector (Cross) Product

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \hat{e}_i \epsilon_{ijk} A_j B_k$$

(i^{th} component)

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$$

e.g.

$$\begin{aligned} (\vec{A} \times \vec{B})_1 &= \sum_{j,k=1}^3 \epsilon_{1jk} A_j B_k \\ &= \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2 \end{aligned}$$



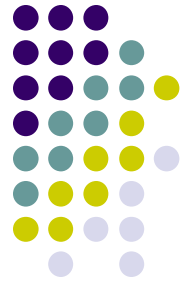
Triple Product (1 of 2)

$$\vec{A} \cdot \vec{B} \times \vec{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B} \quad (\text{cyclic})$$

$$= -\vec{A} \cdot \vec{C} \times \vec{B} = -\vec{C} \cdot \vec{B} \times \vec{A} \quad (\text{non-cyclic})$$

e.g.

$$\begin{aligned} \vec{A} \cdot \vec{B} \times \vec{C} &= A_i (\vec{B} \times \vec{C})_i = A_i \epsilon_{imn} B_m C_n \\ &= C_n \epsilon_{imn} B_m A_i = -C_n \epsilon_{nmi} B_m A_i = -C_n (\vec{B} \times \vec{A})_n = -\vec{C} \cdot \vec{B} \times \vec{A} \end{aligned}$$



Product of ϵ_{ijk}

$$\epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$$

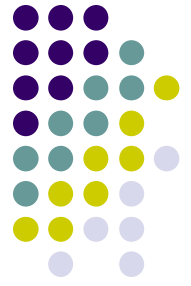
e.g. $\epsilon_{123} \epsilon_{133} = \delta_{23} \delta_{33} - \delta_{23} \delta_{33} = 0$ (no sum)

$$\epsilon_{123} \epsilon_{132} = \delta_{23} \delta_{32} - \delta_{22} \delta_{33} = -1$$

$$\sum_i \epsilon_{i23} \epsilon_{i33} = 0$$
 (sum too)

$$\sum_i \epsilon_{i21} \epsilon_{i32} = 0 = \delta_{23} \delta_{12} - \delta_{22} \delta_{13}$$

$$\sum_i \epsilon_{i21} \epsilon_{i12} = -1 = \delta_{21} \delta_{12} - \delta_{22} \delta_{11}$$



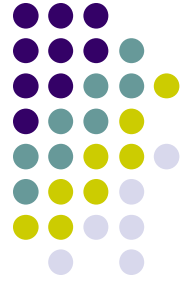
Triple Vector Product

$$\begin{aligned}
 \vec{A} \times (\vec{B} \times \vec{C}) &\cong \varepsilon_{ijk} A_j (\vec{B} \times \vec{C})_k = \varepsilon_{ijk} A_j \varepsilon_{kmn} B_m C_n = \varepsilon_{kij} \varepsilon_{kmn} A_j B_m C_n \\
 &= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) A_j B_m C_n && \text{(for each component)} \\
 &= A_j B_i C_j - A_m B_m C_i = (\vec{A} \cdot \vec{C}) B_i - (\vec{A} \cdot \vec{B}) C_i \\
 &\cong (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}
 \end{aligned}$$

Note:

$$\begin{aligned}
 (\vec{A} \times \vec{B}) \times \vec{C} &= \varepsilon_{ijk} (\varepsilon_{jmn} A_m B_n) C_k = \varepsilon_{jki} \varepsilon_{jmn} A_m B_n C_k \\
 &= (\delta_{km} \delta_{in} - \delta_{kn} \delta_{mi}) A_m B_n C_k \\
 &= A_k B_i C_k - A_i B_n C_n \\
 &= (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{B} \cdot \vec{C}) \vec{A}
 \end{aligned}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$



Vector Calculus - example

$$\nabla \cdot (f\vec{A}) = ?$$

$$\partial_i (f\vec{A})_i = f\partial_i A_i + A_i \partial_i f$$

$$\nabla \cdot (f\vec{A}) = f\nabla \cdot \vec{A} + \vec{A} \cdot \nabla f$$