Advanced Vector Calculus

(optional)

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Scalar (Dot) Product

$$\vec{A} = \sum_{i=1}^{3} A_i \hat{e}_i \equiv A_i \hat{e}_i$$

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

Kronecker Delta Function

= 1 (if i = j)

= 0 (if i ≠ j)

(orthogonal)

$$\vec{A} \cdot \vec{B} = A_i \hat{e}_i \cdot B_j \hat{e}_j = \delta_{ij} A_i B_j$$

$$\vec{A} \cdot \vec{B} = A_i B_i$$

Permutation Symbol ε_{ijk}

ε_{ijk}
= 1 (if ijk are cyclic permutation of 123)
= -1 (if ijk are non-cyclic permutation of 123; interchange a pair)
= 0 (if i, j or k is a duplicated index)

e.g. $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$

$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$

$\varepsilon_{113} = \varepsilon_{221} = \varepsilon_{322} = 0$ …etc

Vector (Cross) Product

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \hat{e}_i \varepsilon_{ijk} A_j B_k$$

(i^{th} component)

$$(\vec{A}\times\vec{B})_i = \varepsilon_{ijk} A_j B_k$$

e.g.

$$(\vec{A}\times\vec{B})_1 = \sum_{j,k=1}^{3} \varepsilon_{1jk} A_j B_k$$

$$= \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2$$

Triple Product (1 of 2)

$$\vec{A}\cdot\vec{B}\times\vec{C}=\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}=\vec{B}\cdot\vec{C}\times\vec{A}=\vec{C}\cdot\vec{A}\times\vec{B}$$

(cyclic)

$$=-\vec{A}\cdot\vec{C}\times\vec{B}=-\vec{C}\cdot\vec{B}\times\vec{A}$$

(non-cyclic)

e.g.

$$\vec{A}\cdot\vec{B}\times\vec{C}=A_i(\vec{B}\times\vec{C})_i=A_i\varepsilon_{imn}B_mC_n$$

$$=C_n\varepsilon_{imn}B_mA_i=-C_n\varepsilon_{nmi}B_mA_i=-C_n(\vec{B}\times\vec{A})_n=-\vec{C}\cdot\vec{B}\times\vec{A}$$

Product of ε_{ijk}

$$\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$$

e.g.

$$\varepsilon_{123}\,\varepsilon_{133} = \delta_{23}\,\delta_{33} - \delta_{23}\,\delta_{33} = 0$$ (no sum)

$$\varepsilon_{123}\,\varepsilon_{132} = \delta_{23}\,\delta_{32} - \delta_{22}\,\delta_{33} = -1$$

$$\sum_i \varepsilon_{i23}\varepsilon_{i33} = 0$$ (sum too)

$$\sum_i \varepsilon_{i21}\varepsilon_{i32} = 0 = \delta_{23}\delta_{12} - \delta_{22}\delta_{13}$$

$$\sum_i \varepsilon_{i21}\varepsilon_{i12} = -1 = \delta_{21}\delta_{12} - \delta_{22}\delta_{11}$$

Triple Vector Product

$$\vec{A}\times(\vec{B}\times\vec{C}) \cong \varepsilon_{ijk}A_j(\vec{B}\times\vec{C})_k = \varepsilon_{ijk}A_j\varepsilon_{kmn}B_mC_n = \varepsilon_{kij}\varepsilon_{kmn}A_jB_mC_n$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})A_jB_mC_n$$

(for each component)

$$= A_jB_iC_j - A_mB_mC_i = (\vec{A}\cdot\vec{C})B_i - (\vec{A}\cdot\vec{B})C_i$$

$$\cong (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

Note:

$$(\vec{A}\times\vec{B})\times\vec{C} = \varepsilon_{ijk}(\varepsilon_{jmn}A_mB_n)C_k = \varepsilon_{jki}\varepsilon_{jmn}A_mB_nC_k$$

$$= (\delta_{km}\delta_{in} - \delta_{kn}\delta_{mi})A_mB_nC_k$$

$$= A_kB_iC_k - A_iB_nC_n$$

$$= (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}$$

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}}$$

Vector Calculus - example

$$\nabla \cdot (f\vec{A}) = ?$$

$$\partial_i (f\vec{A})_i = f\partial_i A_i + A_i \partial_i f$$

$$\nabla \cdot (f\vec{A}) = f\nabla \cdot \vec{A} + \vec{A} \cdot \nabla f$$