$V \Leftrightarrow \vec{E}$

1D example

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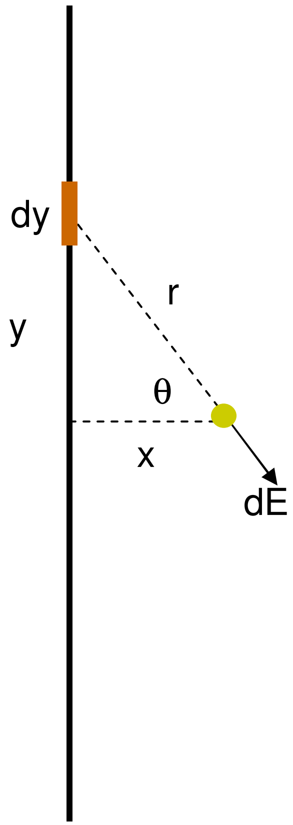
1D uniform charge distribution

At center line of a finite length L,   E = ?



$$|dE| = k\frac{|dq|}{r^2} = k\frac{(\lambda dy)}{x^2+y^2}$$

$$|dE|_x = |dE|\cos\theta = k\frac{(\lambda dy)}{x^2+y^2}\left(\frac{x}{r}\right) = \frac{k\lambda x(dy)}{(x^2+y^2)^{3/2}}$$

$$E = \int_{-L/2}^{L/2}\frac{k\lambda x(dy)}{(x^2+y^2)^{3/2}}$$

$$\int\frac{dy}{r^3} = \int\frac{x\sec^2\theta d\theta}{(x\sec\theta)^3} = \frac{1}{x^2}\int\cos\theta d\theta = \frac{1}{x^2}\sin\theta = \frac{1}{x^2}\left[\frac{y}{\sqrt{x^2+y^2}}\right]$$

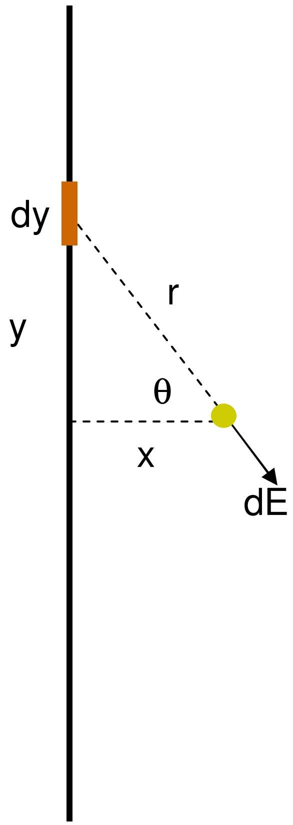
$$\int_{-L/2}^{L/2}\frac{dy}{r^3} = \frac{1}{x^2}\left[\frac{y}{\sqrt{x^2+y^2}}\right]_{-L/2}^{L/2} = \frac{1}{x^2}\frac{L}{\sqrt{x^2+(L/2)^2}} = \frac{1}{x^2}\frac{L}{(L/2)\sqrt{(2x/L)^2+1}} = \frac{2}{x^2}\left(1+\left(\frac{2x}{L}\right)^2\right)^{-1/2}$$

$$E = k\lambda x\cdot\frac{2}{x^2}\left(1+\left(\frac{2x}{L}\right)^2\right)^{-1/2}$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_o x}\left(1+\left(\frac{2x}{L}\right)^2\right)^{-1/2}\hat{x} = \frac{\lambda}{2\pi\varepsilon_o x\sqrt{1+(2x/L)^2}}\hat{x} \qquad \text{(eqn. 1)}$$

Infinite line charge

Infinite length. E = ?



From Eqn. 1, set L → ∞

$$\vec{E} = \lim_{L\to\infty} \frac{\lambda}{2\pi\varepsilon_o x}\left(1+\left(\frac{2x}{L}\right)^2\right)^{-1/2}\hat{x}$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_o x}\hat{x}$$

(eqn. 2)

which can be easily confirmed with Gauss Law

$$EA = \frac{Q_{inside}}{\varepsilon_o}$$

$$E(2\pi rL) = \frac{\lambda L}{\varepsilon_o}$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_o r}\hat{r}$$

point charge

L → 0, E = ?

From Eqn. 1, set L → 0

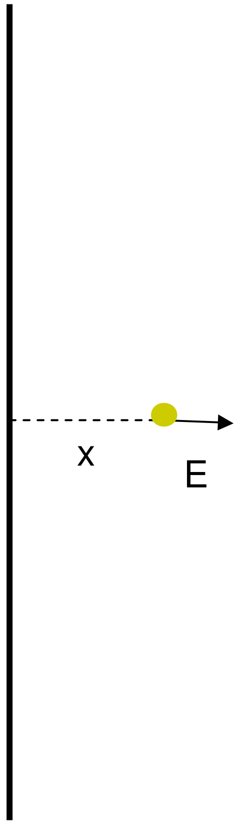
$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_o x\sqrt{1+(2x/L)^2}}\hat{x} = \frac{\lambda L}{2\pi\varepsilon_o x 2x\sqrt{(L/2x)^2+1}}\hat{x}$$

$$\vec{E} = \frac{\lambda L}{4\pi\varepsilon_o x^2}\hat{x} = \frac{Q}{4\pi\varepsilon_o x^2}\hat{x}$$

which is just the point charge equation.

V from E

due to the charges on finite length L. (note: V = 0 at infinity.)



$$\int_0^V dV = -\int_\infty^x \vec{E}\cdot d\vec{\ell}$$

$$V = -\int_\infty^x \frac{\lambda}{2\pi\varepsilon_o x}\left(1+\left(\frac{2x}{L}\right)^2\right)^{-1/2} dx$$

$$u \equiv \frac{2x}{L}$$

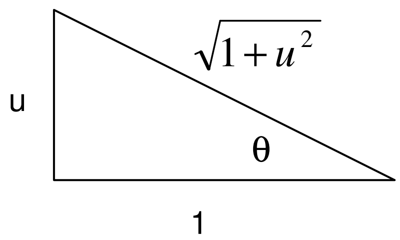
$$V = -\frac{\lambda}{2\pi\varepsilon_o}\int_\infty^u \frac{du}{u\sqrt{1+u^2}}$$

(see next page)

$$V = -\frac{\lambda}{2\pi\varepsilon_o}\ln\left|\frac{u}{1+\sqrt{1+u^2}}\right| = \frac{\lambda}{2\pi\varepsilon_o}\ln\left|\frac{1+\sqrt{1+(2x/L)^2}}{2x/L}\right|$$

(eqn.3)

integral



$$\int_{\infty}^{u} \frac{du}{u\sqrt{1+u^2}} = ?$$

$$u = \tan\theta$$

$$du = \sec^2\theta d\theta$$

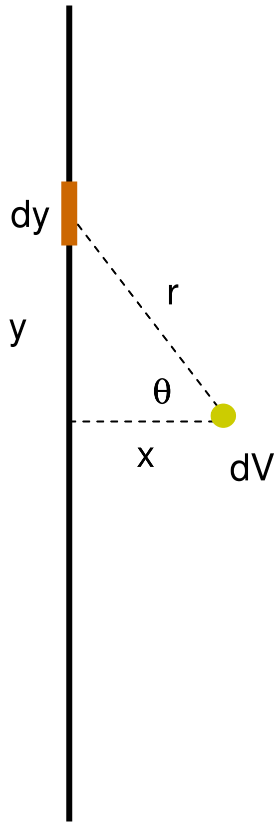
$$\sqrt{1+u^2} = \sec\theta$$

$$\int \frac{du}{u\sqrt{1+u^2}} = \int \frac{\sec^2\theta d\theta}{\tan\theta\sec\theta} = \int \frac{d\theta}{\sin\theta} = \int \csc\theta d\theta = -\ln\left|\csc\theta + \cot\theta\right|$$

$$\int_{\infty}^{u} \frac{du}{u\sqrt{1+u^2}} = -\ln\left|\frac{\sqrt{1+u^2}}{u} + \frac{1}{u}\right|_{\infty}^{u} = \ln\left|\frac{u}{1+\sqrt{1+u^2}}\right|_{\infty}^{u} = \ln\left|\frac{u}{1+\sqrt{1+u^2}}\right|$$

V from integrating pt charges

Finite length L. V = ?



$$dV = k\frac{|dq|}{r} = k\frac{(\lambda dy)}{\sqrt{x^2 + y^2}}$$

$$V = \int_{-L/2}^{L/2} \frac{k\lambda dy}{\sqrt{x^2 + y^2}}$$

$$\int \frac{dy}{r} = \int \frac{x\sec^2\theta d\theta}{(x\sec\theta)} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| = \ln\left|\frac{r}{x} + \frac{y}{x}\right|$$

$$\int_{-L/2}^{L/2} \frac{dy}{r} = \ln\left|\frac{\sqrt{x^2+y^2}+y}{x}\right|_{-L/2}^{L/2} = \ln\left|\frac{\sqrt{x^2+(L/2)^2}+L/2}{\sqrt{x^2+(L/2)^2}-L/2}\right| = \ln\left|\frac{\sqrt{(2x/L)^2+1}+1}{\sqrt{(2x/L)^2+1}-1}\right|$$

$$V = \frac{\lambda}{4\pi\varepsilon_o}\ln\left|\frac{\sqrt{(2x/L)^2+1}+1}{\sqrt{(2x/L)^2+1}-1}\right| \qquad \text{(eqn. 4)}$$

Note: Equations 3 & 4 are essentially the same !!!!!

E from V – finite linear charge

From eqn (3)

$$V = \frac{\lambda}{2\pi\varepsilon_o}\ln\left|\frac{1+\sqrt{1+(2x/L)^2}}{2x/L}\right| = \frac{\lambda}{2\pi\varepsilon_o}\left[\ln\left|1+\sqrt{1+u^2}\right| - \ln|u|\right]$$

$$u = 2x/L$$

$$E = -\frac{dV}{dx} = -\frac{2}{L}\frac{dV}{du} = -\frac{\lambda}{\pi\varepsilon_o L}\left[\frac{\frac{1}{2}(1+u^2)^{-1/2}(2u)}{1+\sqrt{1+u^2}} - \frac{1}{u}\right] = -\frac{\lambda}{\pi\varepsilon_o L}\left[\frac{u^2-\sqrt{1+u^2}\left(1+\sqrt{1+u^2}\right)}{u\sqrt{1+u^2}\left(1+\sqrt{1+u^2}\right)}\right]$$

$$\vec{E} = -\hat{x}\frac{\lambda}{\pi\varepsilon_o L}\left[\frac{u^2-\sqrt{1+u^2}-1-u^2}{u\sqrt{1+u^2}\left(1+\sqrt{1+u^2}\right)}\right] = \hat{x}\frac{\lambda}{\pi\varepsilon_o L u\sqrt{1+u^2}} = \frac{\lambda}{2\pi\varepsilon_o x\sqrt{1+(2x/L)^2}}\hat{x}$$

From eqn (4)

$$V = \frac{\lambda}{4\pi\varepsilon_o}\ln\left|\frac{\sqrt{(2x/L)^2+1}+1}{\sqrt{(2x/L)^2+1}-1}\right| = \frac{\lambda}{4\pi\varepsilon_o}\left[\ln\left|\sqrt{u^2+1}+1\right| - \ln\left|\sqrt{u^2+1}-1\right|\right]$$

$$\vec{E} = -\hat{x}\frac{dV}{dx} = -\hat{x}\frac{2}{L}\frac{dV}{du} = -\hat{x}\frac{\lambda}{2\pi\varepsilon_o L}\left[\frac{\frac{1}{2}(u^2+1)^{-1/2}(2u)}{\sqrt{u^2+1}+1} - \frac{\frac{1}{2}(u^2+1)^{-1/2}(2u)}{\sqrt{u^2+1}-1}\right]$$

$$\vec{E} = -\hat{x}\frac{\lambda}{2\pi\varepsilon_o L}\frac{u}{\sqrt{u^2+1}}\left[\frac{\left(\sqrt{u^2+1}-1\right)-\left(\sqrt{u^2+1}+1\right)}{u^2+1-1}\right] = \hat{x}\frac{2\lambda}{2\pi\varepsilon_o L u\sqrt{u^2+1}}$$

$$\vec{E} = \hat{x}\frac{\lambda}{2\pi\varepsilon_o x\sqrt{1+(2x/L)^2}}$$

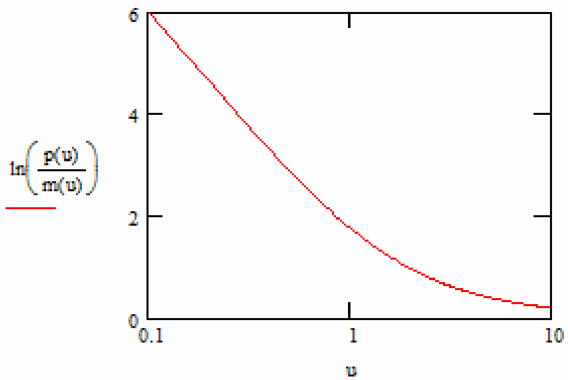
Both give the same E as in eqn (1)

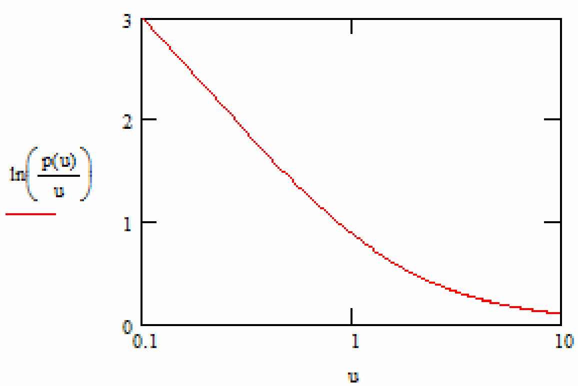
Eqn(3) is the same as Eqn(4)

$u := 0.1, 0.15 .. 10$

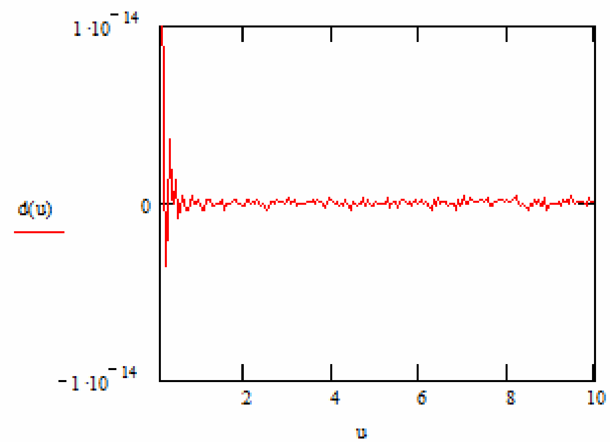
$p(u) := \sqrt{1 + u^2} + 1$

$m(u) := \sqrt{1 + u^2} - 1$



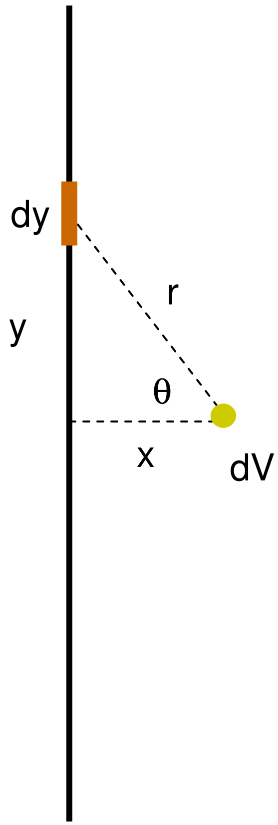


$d(u) := \ln\left(\frac{p(u)}{m(u)}\right) - 2 \cdot \ln\left(\frac{p(u)}{u}\right)$



V from infinite line charge

From eqn (4)



$$V = \frac{\lambda}{4\pi\varepsilon_o} \ln \left| \frac{\sqrt{(2x/L)^2 + 1} + 1}{\sqrt{(2x/L)^2 + 1} - 1} \right|$$

$$L \to \infty$$

$$\sqrt{(2x/L)^2 + 1} = \left(1 + (2x/L)^2\right)^{1/2} \approx 1 + 2\left(\frac{x}{L}\right)^2$$

$$\ln \left| \frac{\sqrt{(2x/L)^2 + 1} + 1}{\sqrt{(2x/L)^2 + 1} - 1} \right| \approx \ln \left| \frac{2}{2(x/L)^2} \right| = 2\ln\left|\frac{L}{x}\right| = -2\ln|x| + C$$

$$V = -\frac{\lambda}{2\pi\varepsilon_o} \ln|x| + C$$

(eqn. 5)

even though the constant ⟶ ∞

V from E: Infinite line charge

From eqn (3)

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_o x}\hat{x}$$

$$V = -\int \vec{E} \cdot d\vec{\ell} = -\frac{\lambda}{2\pi\varepsilon_o}\int \frac{dx}{x} = -\frac{\lambda}{2\pi\varepsilon_o}\ln|x| + C$$

which is the same as eqn (5)

V of a point charge

From eqn (4), set L → 0

$$V=\frac{\lambda}{4\pi\varepsilon_o}\ln\left|\frac{\sqrt{(2x/L)^2+1}+1}{\sqrt{(2x/L)^2+1}-1}\right|=\frac{\lambda}{4\pi\varepsilon_o}\ln\left|\frac{\sqrt{1+(L/2x)^2}+(L/2x)}{\sqrt{1+(L/2x)^2}-(L/2x)}\right|\approx\frac{\lambda}{4\pi\varepsilon_o}\ln\left|\frac{1+(L/2x)}{1-(L/2x)}\right|$$

$$\ln|1+x|=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\approx x$$

$$\ln\left|\frac{1+(L/2x)}{1-(L/2x)}\right|=\ln\left|1+\frac{2(L/2x)}{1-(L/2x)}\right|\approx\frac{2(L/2x)}{1-(L/2x)}=\frac{2L}{2x-L}\approx\frac{L}{x}$$

$$V=\frac{\lambda}{4\pi\varepsilon_o}\left(\frac{L}{x}\right)=\frac{Q}{4\pi\varepsilon_o x}$$

which is just the point charge equation.