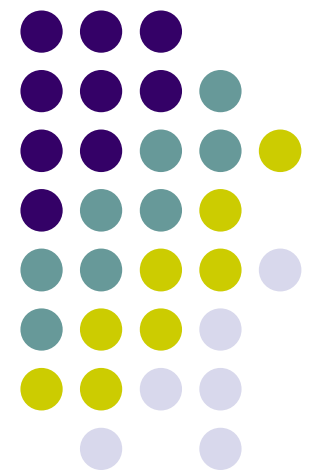


Chapter 3

Vector Calculus

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Del - operator ∇

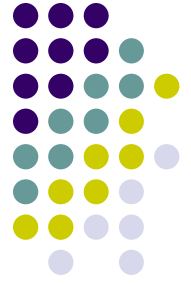
Gradient $\nabla f = \hat{e}_i \partial_i f = \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z}$

Divergence $\nabla \cdot \vec{A} = \partial_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

Curl $\nabla \times \vec{A} = \hat{e}_i \varepsilon_{ijk} \partial_j A_k = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

Laplacian $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

can operate on scalar or vector



2 useful identities

$$\nabla \cdot \nabla \times \vec{A} = 0$$

for any vector function \mathbf{A}

$$\nabla \times \nabla f = 0$$

for any scalar function f

$$\nabla \cdot \nabla \times \vec{A} = \begin{vmatrix} \partial / \partial x_1 & \partial / \partial x_2 & \partial / \partial x_3 \\ \partial / \partial x_1 & \partial / \partial x_2 & \partial / \partial x_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = 0$$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial / \partial x_1 & \partial / \partial x_2 & \partial / \partial x_3 \\ \partial f / \partial x_1 & \partial f / \partial x_2 & \partial f / \partial x_3 \end{vmatrix} = 0$$

$$\begin{array}{ll} \text{in static fields...} & \nabla \cdot \vec{B} = 0 \\ & \nabla \times \vec{E} = 0 \end{array} \quad \longrightarrow \quad \begin{array}{l} \vec{B} = \nabla \times \vec{A} \\ \vec{E} = -\nabla V \end{array}$$



Differential - Cartesian

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$



Differential - Cylindrical

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}} \frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}} \frac{1}{r} \frac{\partial V}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & r A_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{\mathbf{z}} \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\phi) - \frac{\partial A_r}{\partial \phi} \right]$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$



Differential - Spherical

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}} \frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}} \frac{1}{R} \frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} R & \hat{\boldsymbol{\phi}} R \sin \theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & R A_\theta & (R \sin \theta) A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}} \frac{1}{R \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] + \hat{\boldsymbol{\theta}} \frac{1}{R} \left[\frac{1}{\sin \theta} \frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R} (R A_\phi) \right] + \hat{\boldsymbol{\phi}} \frac{1}{R} \left[\frac{\partial}{\partial R} (R A_\theta) - \frac{\partial A_R}{\partial \theta} \right]$$

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$

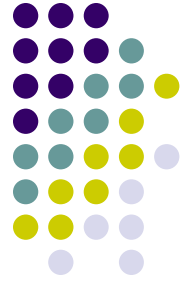


Homework

Show that:

1. $\nabla \cdot \nabla \times \vec{A} = 0$ in spherical coordinates.
2. $\nabla^2 (fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$ for any scalar function f & g .
3. $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ for any vector function \mathbf{A} .
4. $\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B}$
5. $\nabla \cdot (\vec{E} \times \vec{H}) = ?$

Also, do Ch. 3 # 4, 9, 12, 15, 18



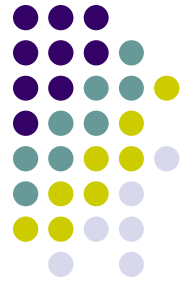
Group exercise

(1)

(2)

$\nabla \cdot (\vec{E} \times \vec{H}) = ?$ What are α & β ??

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla) \vec{B}$$



Exercise -2

$$\vec{A} = x\hat{y} - xy\hat{z}$$

$$\vec{B} = 3\hat{x} + 4x^2y^2\hat{y}$$

$$\vec{B} - 2\vec{A} = ?$$

$$\vec{A} \cdot \vec{B} = ?$$

$$\vec{A} \times \vec{B} = ?$$

$$\nabla \cdot \vec{A} = ?$$

$$\nabla \cdot \vec{B} = ?$$

$$\nabla \times \vec{A} = ?$$

$$\nabla \times \vec{B} = ?$$