Method of Virtual Work
Beam Deflection Example

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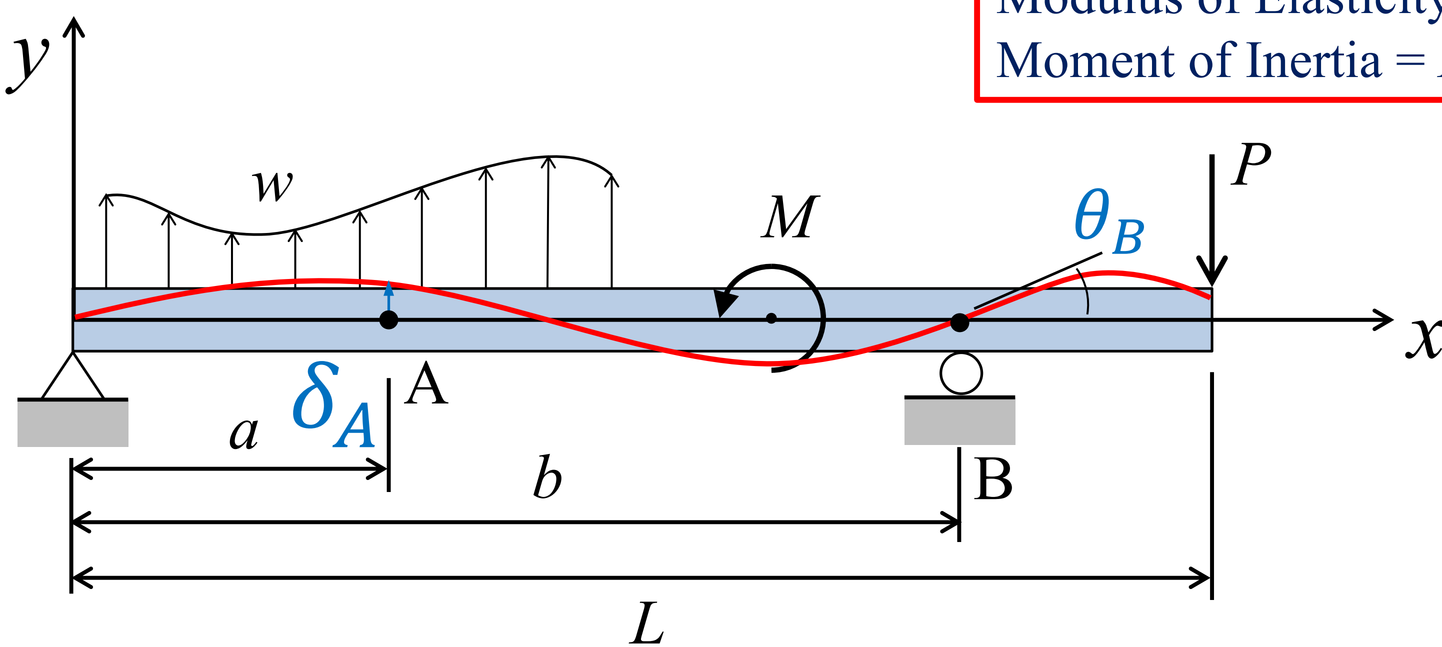
San Jose State University

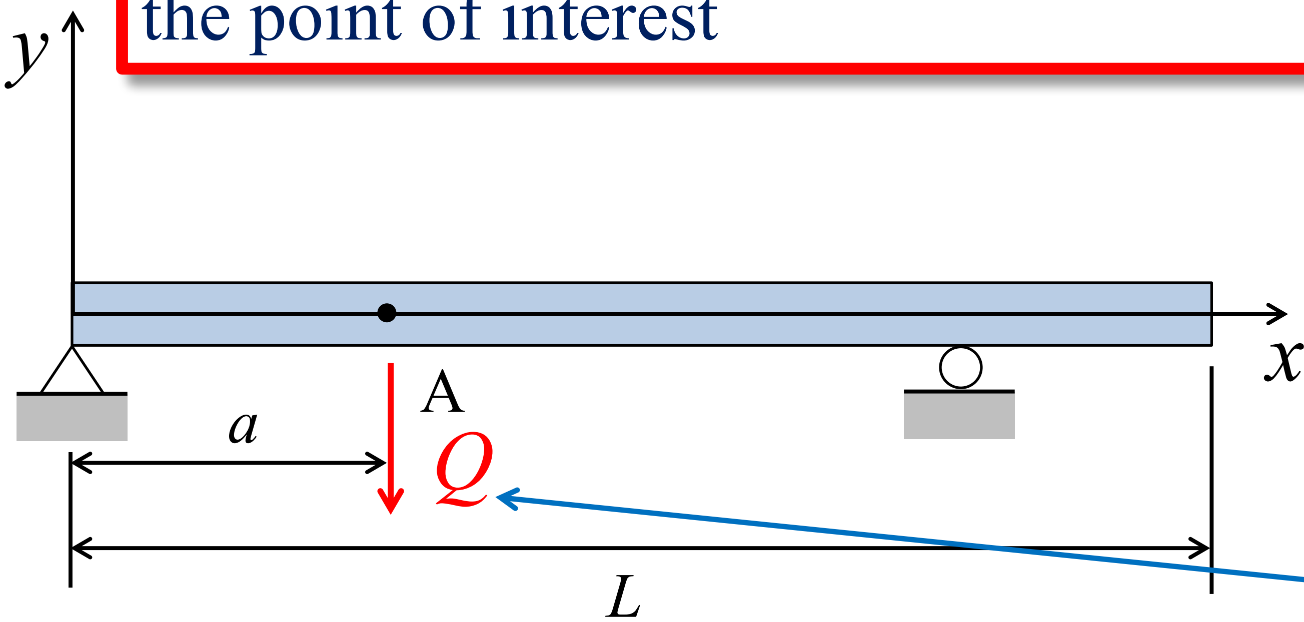
Summary of Procedure for Finding Bending Deformation Using Virtual Work

Modulus of Elasticity = E
Moment of Inertia = I



We want to find the deflection at point A and the slope at point B due to the applied loads

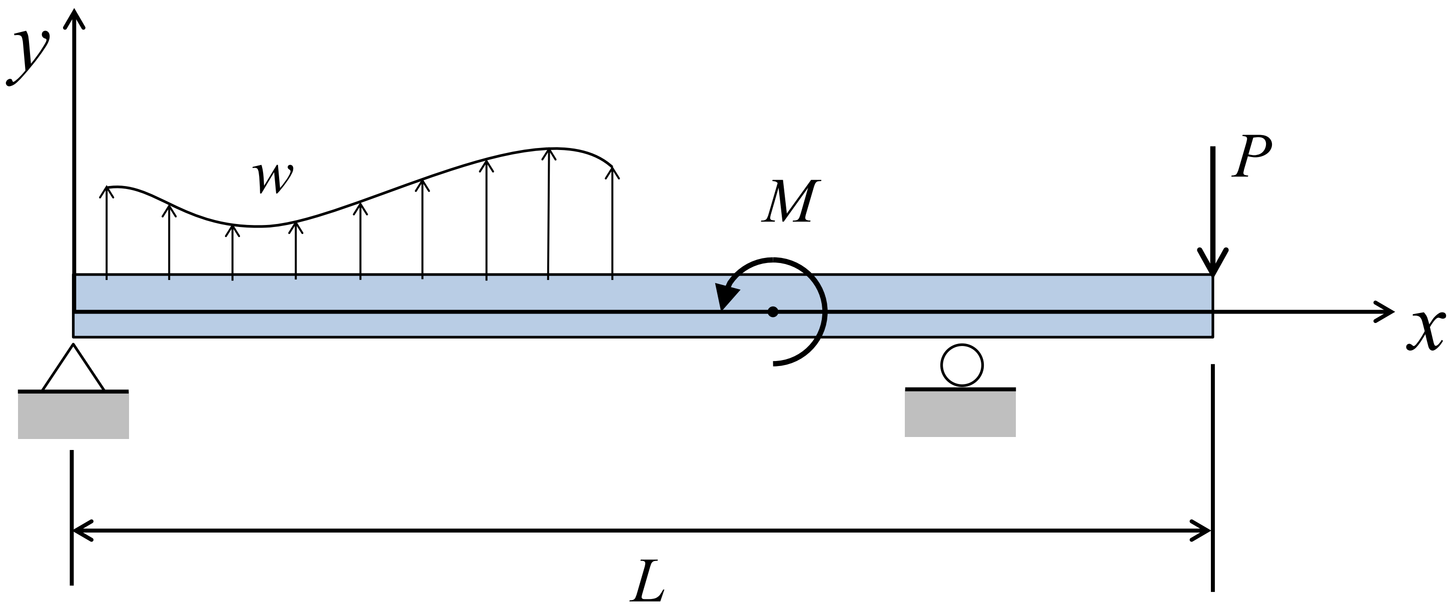
Step 1 – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest



Convenient to set $Q = 1$

From an equilibrium analysis, find the internal bending moment function for the virtual system: $M_Q(x)$

Step 2 – Replace all of the loads on the structure and perform the real analysis



From an equilibrium analysis, find the internal bending moment function for the real system: $M_P(x)$

Step 3 – Evaluate the virtual work product integrals and solve for the deformation of interest

$$Q\delta_A = \int_0^L M_Q \frac{M_P}{EI} dx$$

If the bending stiffness, *EI*, is constant:

$$Q\delta_A = \frac{1}{EI}\int_0^L M_Q M_P dx$$

Table in textbook appendix is provided to help evaluate product integrals of this type

Table to Evaluate Virtual Work Product Integrals

Appendix Table.2

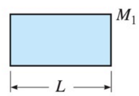
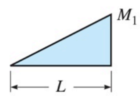
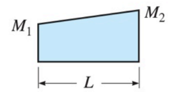
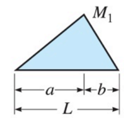
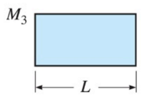
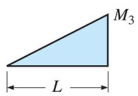
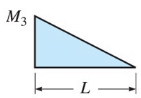
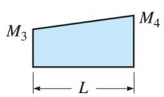
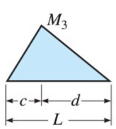
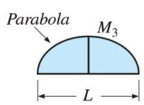
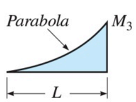
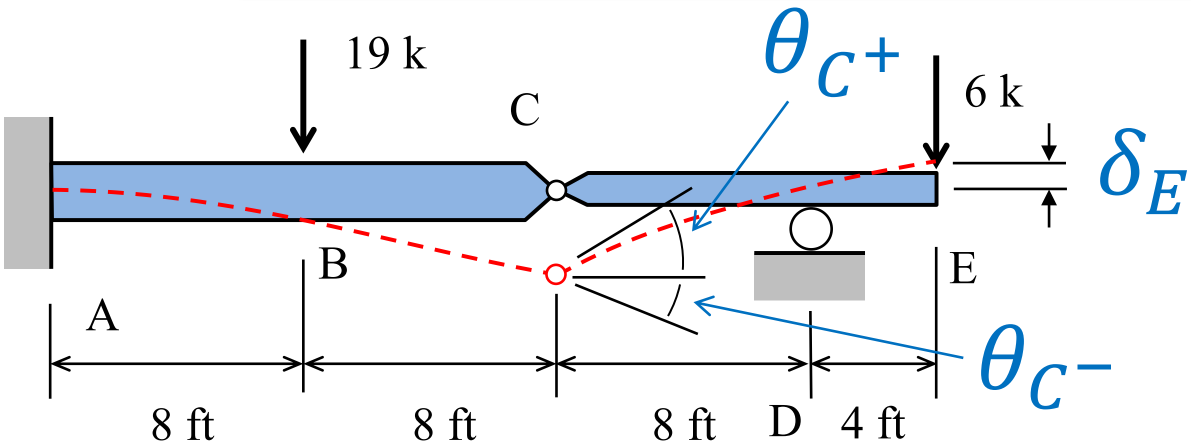
M_Q \ M_P				
	M_1M_3L	$\frac{1}{2}M_1M_3L$	$\frac{1}{2}(M_1+M_2)M_3L$	$\frac{1}{2}M_1M_3L$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{3}M_1M_3L$	$\frac{1}{6}(M_1+2M_2)M_3L$	$\frac{1}{6}M_1M_3(L+a)$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{6}M_1M_3L$	$\frac{1}{6}(2M_1+M_2)M_3L$	$\frac{1}{6}M_1M_3(L+b)$
	$\frac{1}{2}M_1(M_3+M_4)L$	$\frac{1}{6}M_1(M_3+2M_4)L$	$\frac{1}{6}M_1(2M_3+M_4)L$ $+\frac{1}{6}M_2(M_3+2M_4)L$	$\frac{1}{6}M_1M_3(L+b)$ $+\frac{1}{6}M_1M_4(L+a)$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{6}M_1M_3(L+c)$	$\frac{1}{6}M_1M_3(L+d)$ $+\frac{1}{6}M_2M_3(L+c)$	for $c \le a$: $\left(\frac{1}{3}-\frac{(a-c)^2}{6ad}\right)M_1M_3L$
	$\frac{2}{3}M_1M_3L$	$\frac{1}{3}M_1M_3L$	$\frac{1}{3}(M_1+M_2)M_3L$	$\frac{1}{3}M_1M_3\left(L+\frac{ab}{L}\right)$
	$\frac{1}{3}M_1M_3L$	$\frac{1}{4}M_1M_3L$	$\frac{1}{12}(M_1+3M_2)M_3L$	$\frac{1}{12}M_1M_3\left(3a+\frac{a^2}{L}\right)$

Table is as useful tool to evaluate product integrals of the form:

$$\int_0^L M_Q\, M_P\, dx$$

Beam Deflection Example



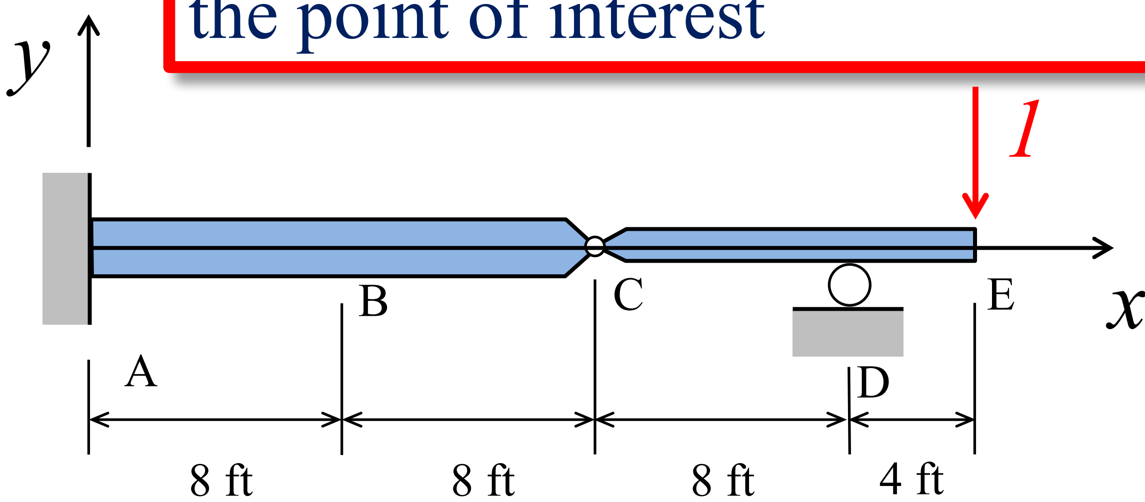
The overhanging beam shown has a fixed support at A, a roller support at C and an internal hinge at B. EI_{ABC} = 2,000,000 k-in^2 and EI_{CDE} = 800,000 k-in^2

For the loads shown, find the following:

1. The vertical deflection at point E;
2. The slope just to the left of the internal hinge at C;
3. The slope just to the right of the internal hinge at C

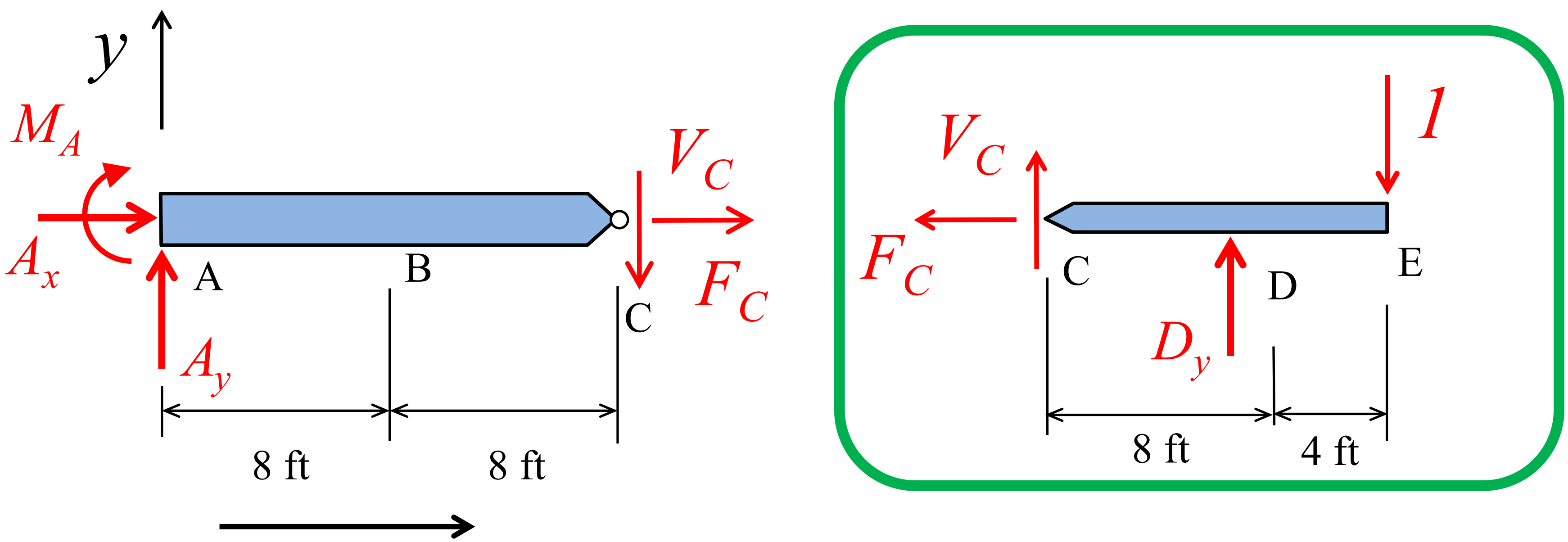
Find the Deflection at Point E

Step 1 – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest



From an equilibrium analysis, find the internal bending moment function for the virtual system: $M_Q(x)$

Find the Moment Diagram for the Virtual System



$$\circlearrowleft + \sum M_A = 0 \rightarrow M_A = 8 \text{ ft}$$

$$\xrightarrow{+} \sum F_x = 0 \rightarrow A_x = 0$$

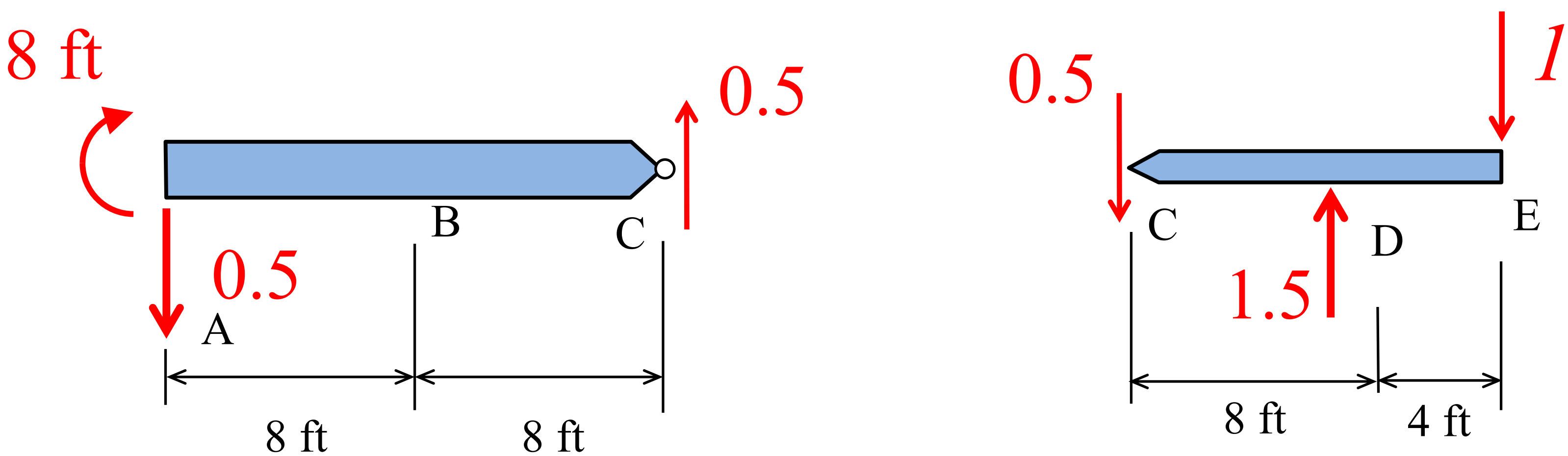
$$+\uparrow \sum F_y = 0 \rightarrow A_y = -0.5$$

$$\circlearrowleft + \sum M_C = 0 \rightarrow D_y = 1.5$$

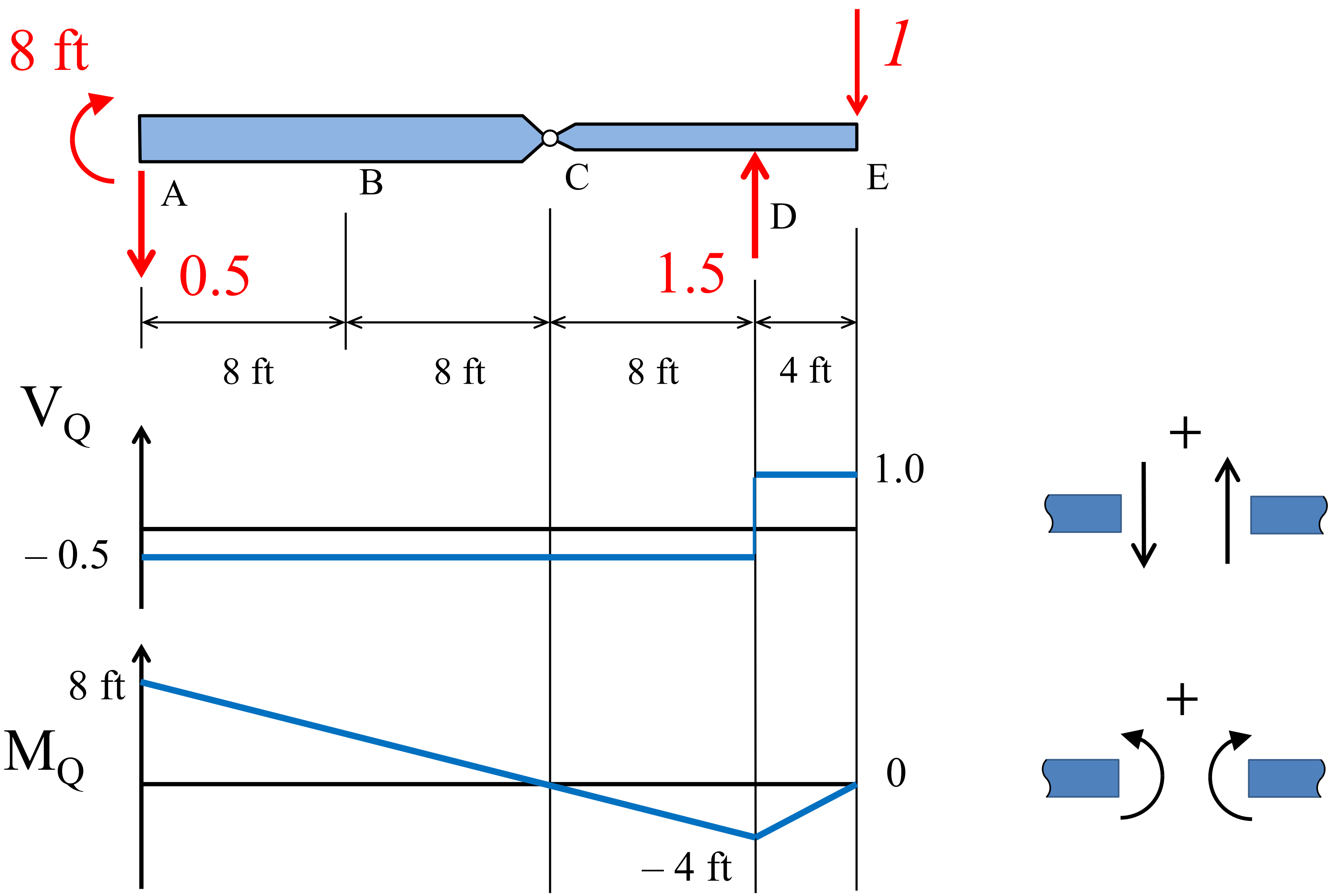
$$\xrightarrow{+} \sum F_x = 0 \rightarrow F_B = 0$$

$$+\uparrow \sum F_y = 0 \rightarrow V_C = -0.5$$

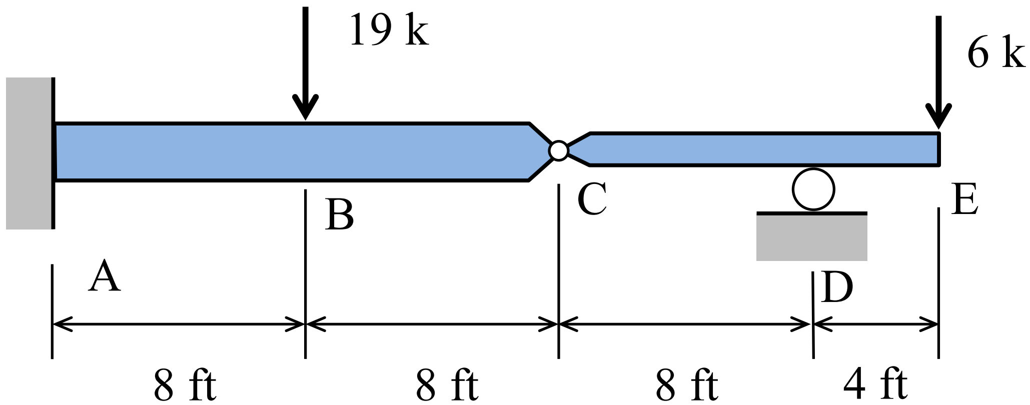
Support Reactions for the Virtual System



Moment Diagram for the Virtual System

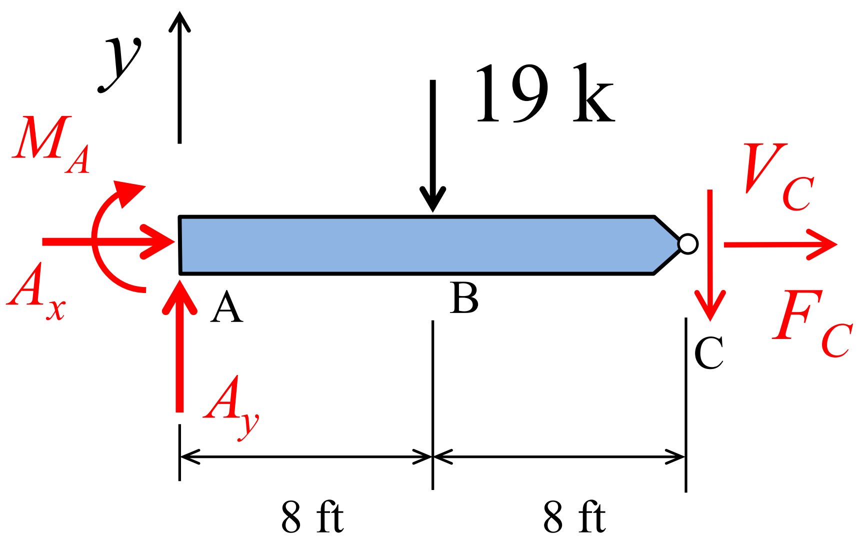


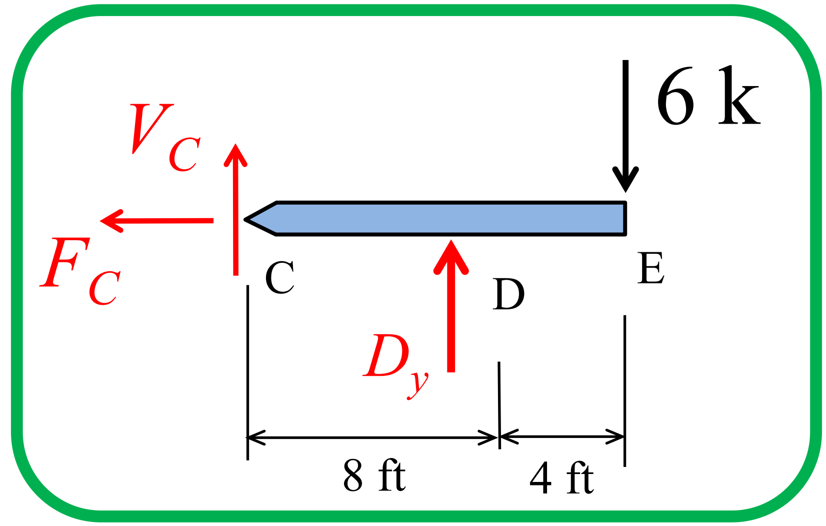
Step 2 – Replace all of the loads on the structure and perform the real analysis



From an equilibrium analysis, find the internal bending moment function for the real system: $M_P(x)$

Find the Moment Diagram for the Real System





$\circlearrowleft + \sum M_A = 0 \rightarrow$ $M_A = -104$ k-ft

$\overset{+}{\rightarrow} \sum F_x = 0 \rightarrow$ $A_x = 0$

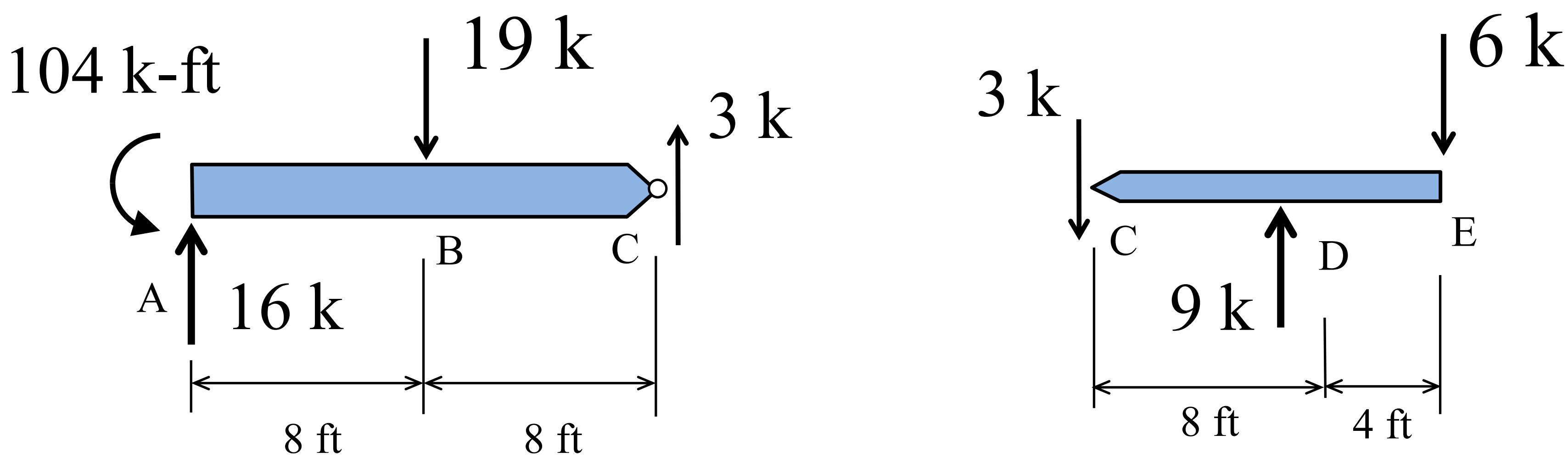
$+\uparrow \sum F_y = 0 \rightarrow$ $A_y = 16$ k

$\circlearrowleft + \sum M_C = 0 \rightarrow$ $D_y = 9$ k

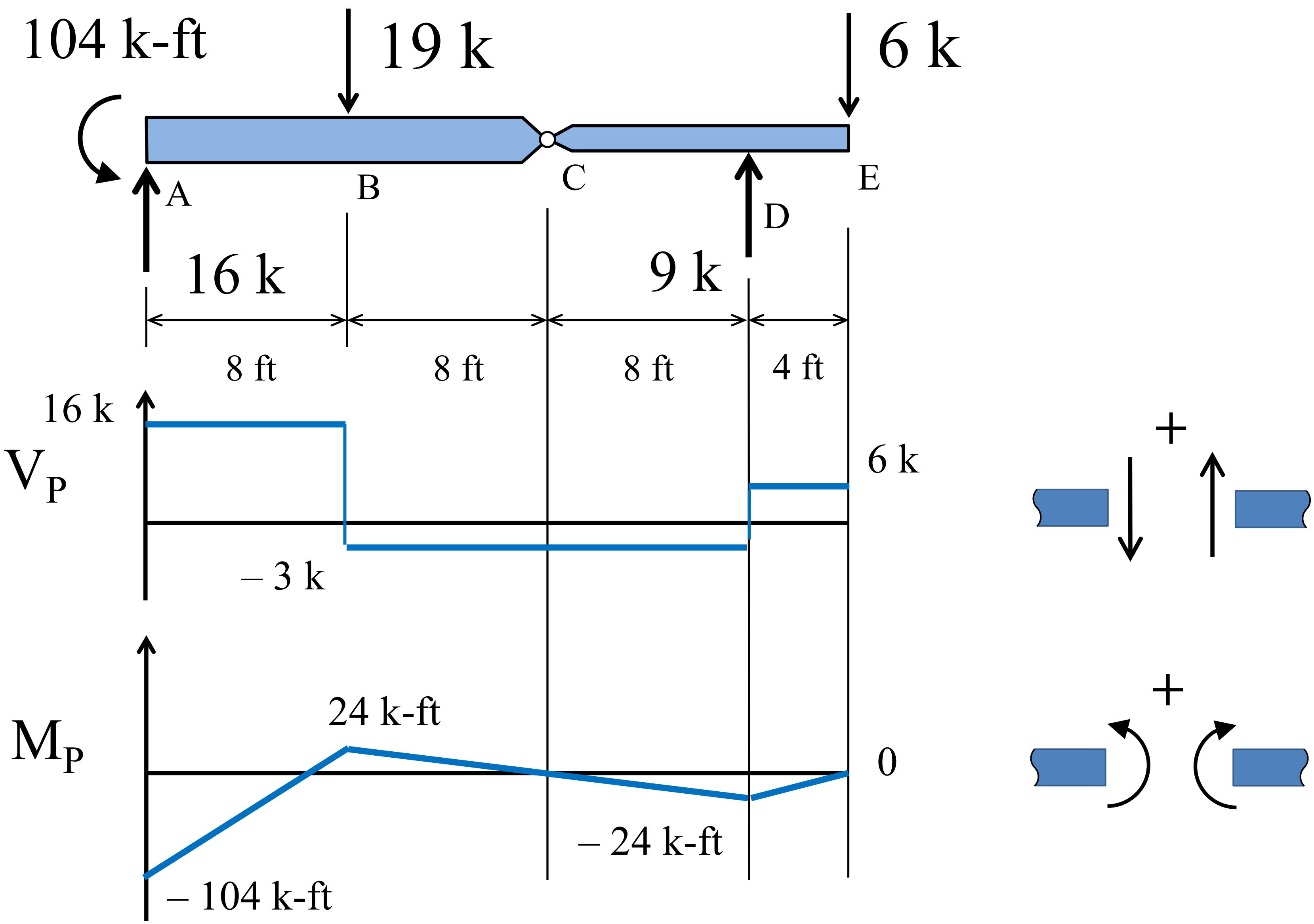
$\overset{+}{\rightarrow} \sum F_x = 0 \rightarrow$ $F_B = 0$

$+\uparrow \sum F_y = 0 \rightarrow$ $V_C = -3$ k

Support Reactions for the Real System



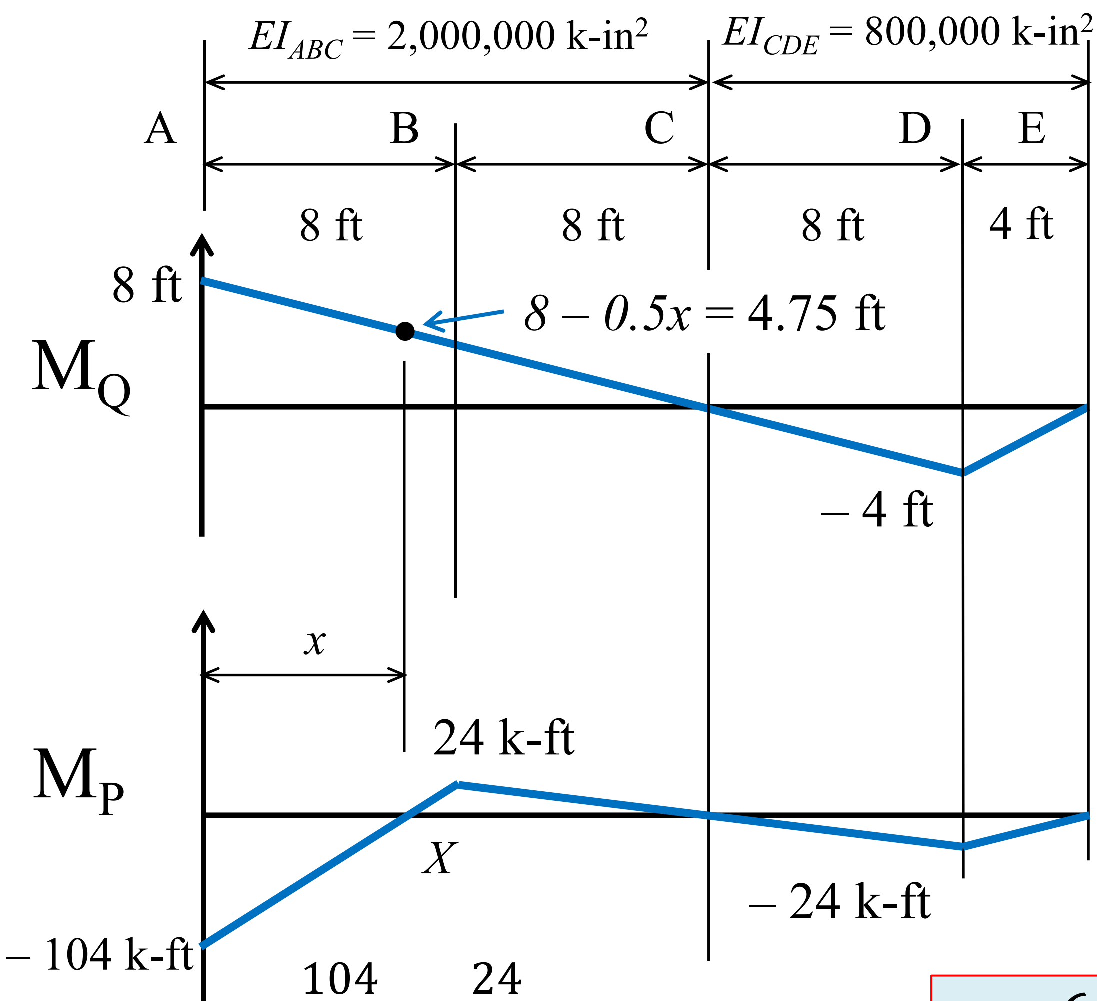
Moment Diagram for the Real System

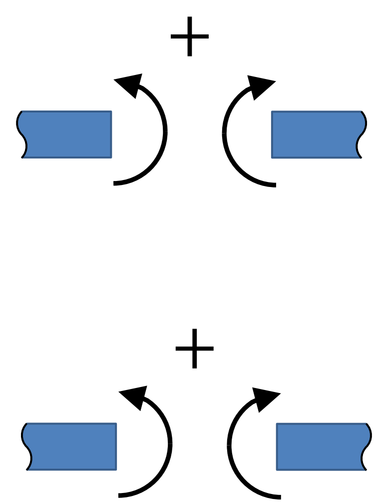


Step 3 – Evaluate the virtual work product integrals and solve for the deformation of interest

$$1 \cdot \delta_E = \frac{1}{EI} \int_0^L M_Q \, M_P dx$$

Use Table to evaluate product integrals

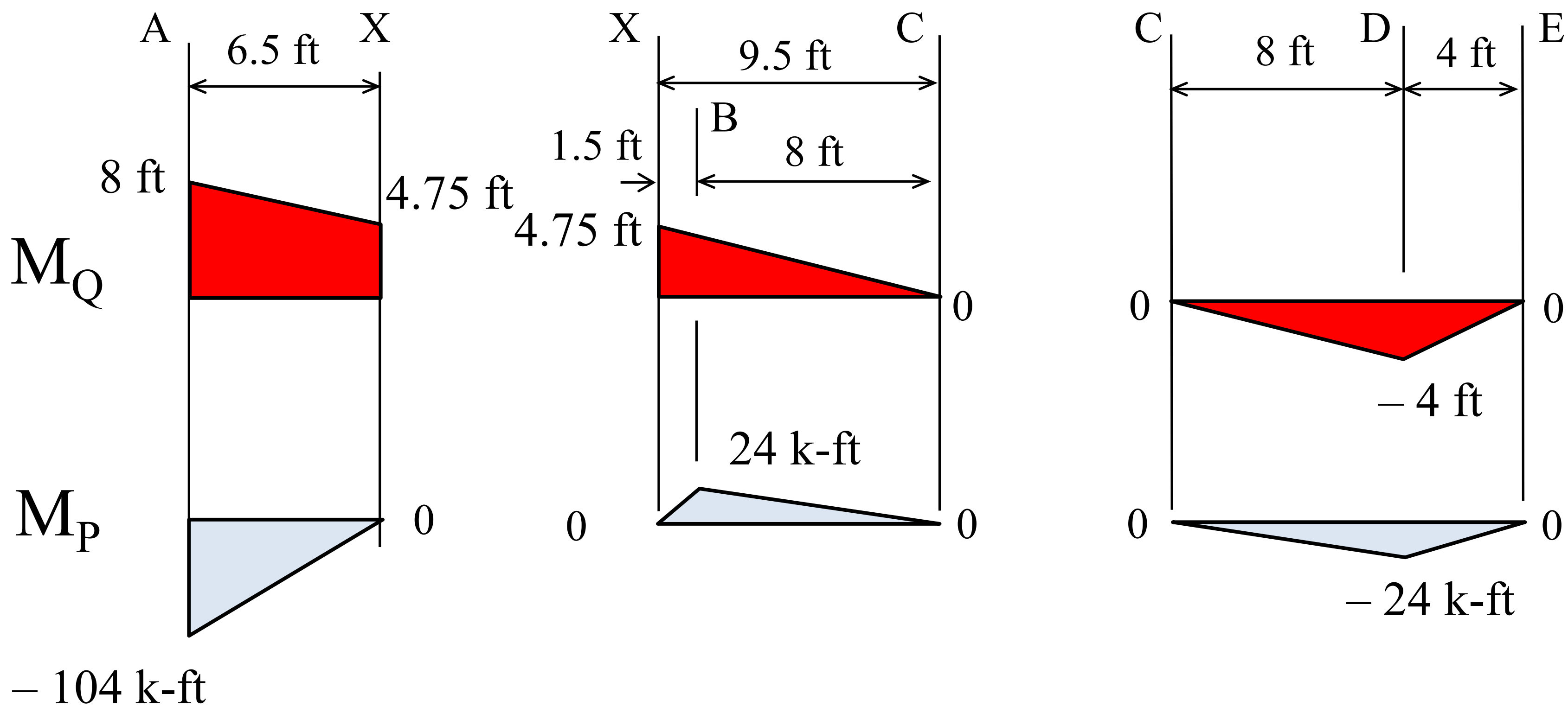




$$\frac{104}{x} = \frac{24}{8 - x} \qquad 832 = 128x$$

$$x = 6.5 \text{ ft}$$

Evaluate Product Integrals



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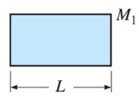
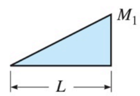
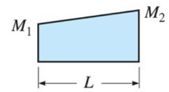
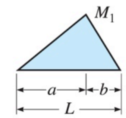
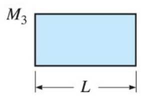
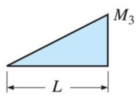
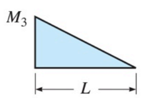
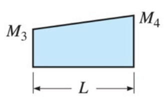
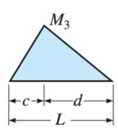
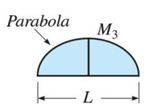
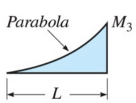
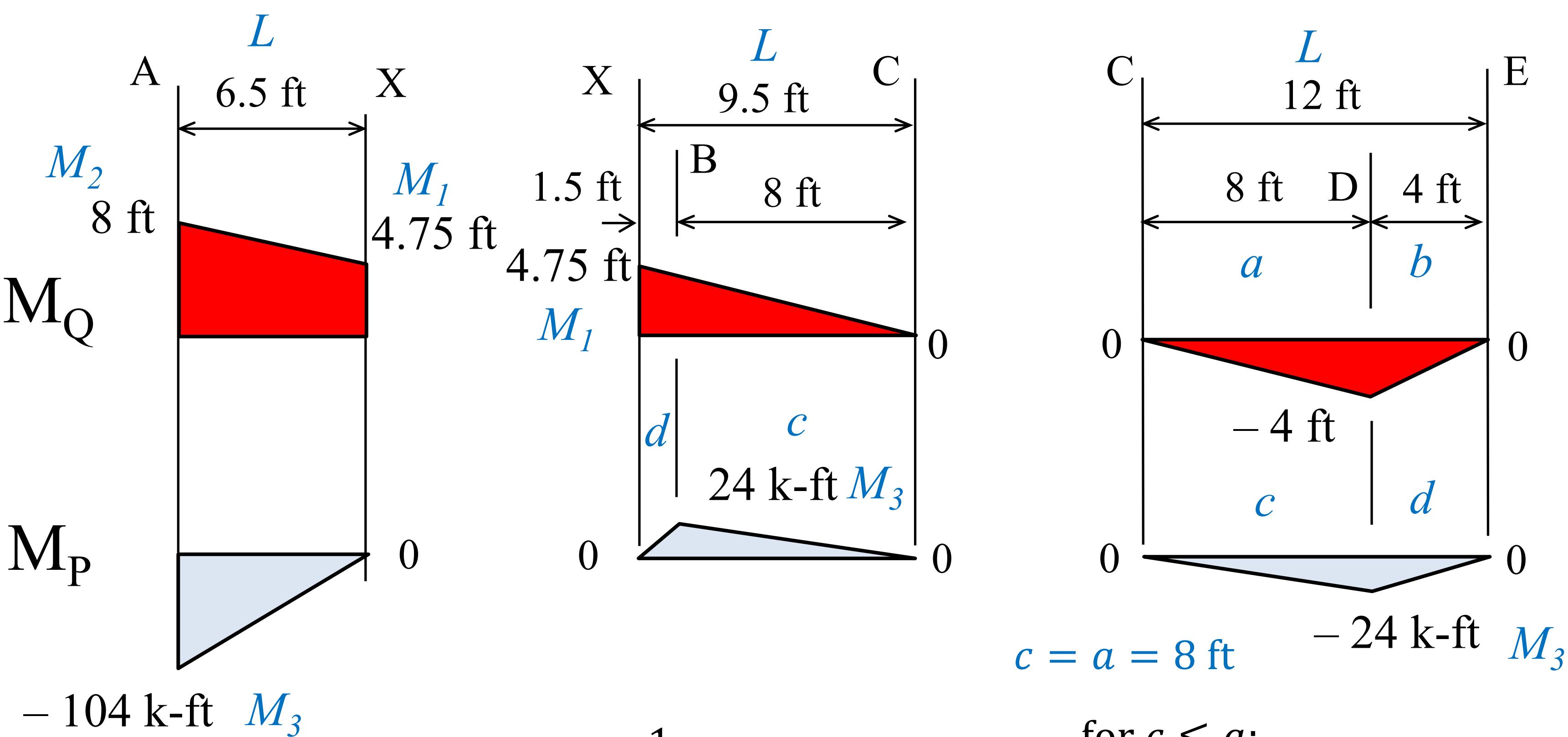
M_Q / M_P				
	M_1M_3L	$\frac{1}{2}M_1M_3L$	$\frac{1}{2}(M_1+M_2)M_3L$	$\frac{1}{2}M_1M_3L$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{3}M_1M_3L$	AX $\frac{1}{6}(M_1+2M_2)M_3L$	$\frac{1}{6}M_1M_3(L+a)$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{6}M_1M_3L$	$\frac{1}{6}(2M_1+M_2)M_3L$	$\frac{1}{6}M_1M_3(L+b)$
	$\frac{1}{2}M_1(M_3+M_4)L$	$\frac{1}{6}M_1(M_3+2M_4)L$	$\frac{1}{6}M_1(2M_3+M_4)L + \frac{1}{6}M_2(M_3+2M_4)L$	$\frac{1}{6}M_1M_3(L+b) + \frac{1}{6}M_1M_4(L+a)$
	$\frac{1}{2}M_1M_3L$	XC $\frac{1}{6}M_1M_3(L+c)$	$\frac{1}{6}M_1M_3(L+d) + \frac{1}{6}M_2M_3(L+c)$	CE for $c \le a$: $\left(\frac{1}{3}-\frac{(a-c)^2}{6ad}\right)M_1M_3L$
	$\frac{2}{3}M_1M_3L$	$\frac{1}{3}M_1M_3L$	$\frac{1}{3}(M_1+M_2)M_3L$	$\frac{1}{3}M_1M_3\left(L+\frac{ab}{L}\right)$
	$\frac{1}{3}M_1M_3L$	$\frac{1}{4}M_1M_3L$	$\frac{1}{12}(M_1+3M_2)M_3L$	$\frac{1}{12}M_1M_3\left(3a+\frac{a^2}{L}\right)$

Table is as useful tool to evaluate product integrals of the form:

$$\int_0^L M_Q\, M_P\, dx$$

Evaluate Product Integrals



$$\frac{1}{6}(M_1 + 2M_2)M_3L$$

$$\frac{1}{6}\big(4.75 + 2(8)\big)(-104)(6.5)$$

– 2337.83 k-ft^3

$$\frac{1}{6}M_1M_3(L + c)$$

$$\frac{1}{6}(4.75)(24)(9.5 + 8)$$

332.5 k-ft^3

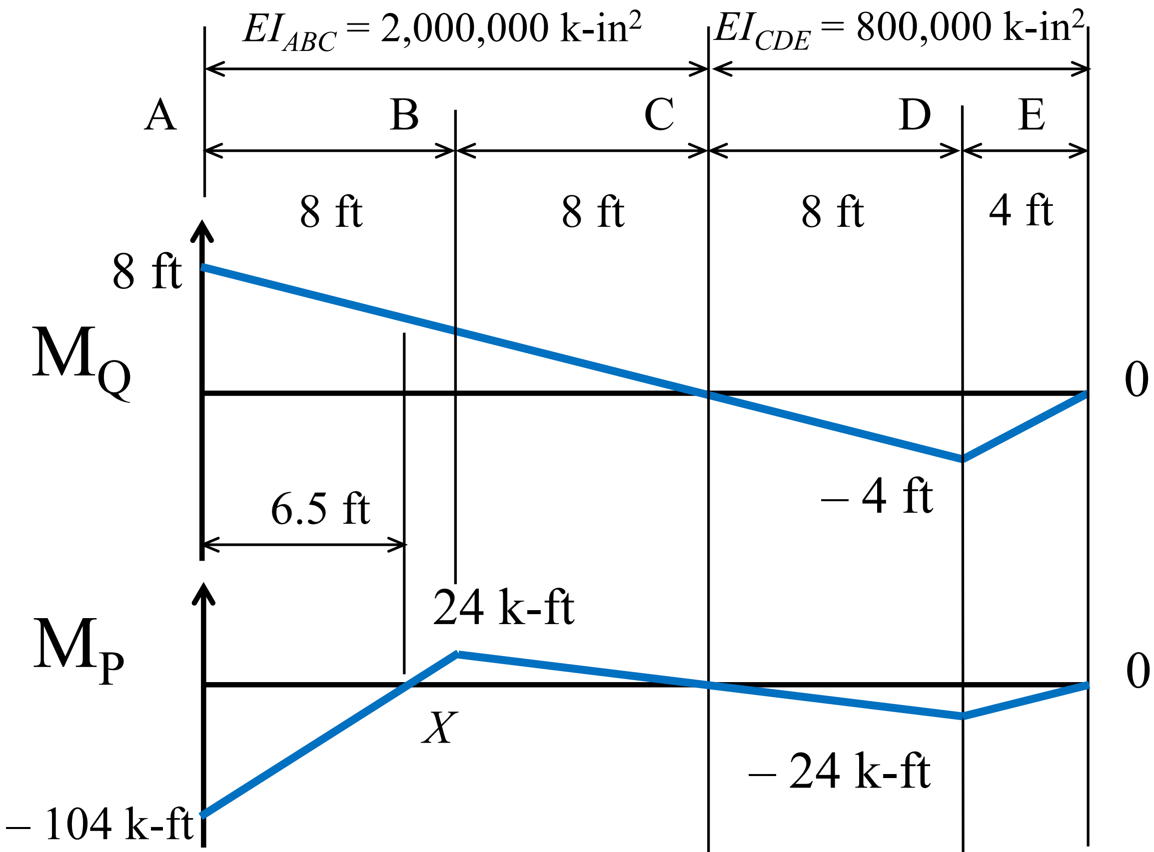
for $c \leq a$:

$$\left(\frac{1}{3} - \frac{(a-c)^2}{6ad}\right)M_1M_3L$$

$$\frac{1}{3}(-4)(-24)(12)$$

384 k-ft^3

Evaluate Product Integrals



$$1 \cdot \delta_E = \frac{1}{EI} \int_0^L M_Q \, M_P dx$$

Segment AX

– 2337.83 k-ft³

Segment XC

332.5 k-ft³

Segment CDE

384 k-ft³

$$\int_0^{L_{ABC}} M_Q \, M_P dx = (-2337.83 + 332.5 \text{ k}-\text{ft}^3)\left(\frac{12^3 \text{ in}^3}{\text{ft}^3}\right) = -3{,}465{,}216.0 \text{ k}-\text{in}^3$$

$$\int_0^{L_{CDE}} M_Q \, M_P dx = (384 \text{ k}-\text{ft}^3)\left(\frac{12^3 \text{ in}^3}{\text{ft}^3}\right) = 663{,}552 \text{ k}-\text{in}^3$$

Evaluate Product Integrals

$$\int_0^{L_{ABC}} M_Q\, M_P dx = \left(-2337.83 + 332.5 \text{ k}-\text{ft}^3\right)\left(\frac{12^3 \text{ in}^3}{\text{ft}^3}\right) = -3{,}465{,}216.0 \text{ k}-\text{in}^3$$

$$\int_0^{L_{CDE}} M_Q\, M_P dx = \left(384 \text{ k}-\text{ft}^3\right)\left(\frac{12^3 \text{ in}^3}{\text{ft}^3}\right) = 663{,}552 \text{ k}-\text{in}^3$$

$$1 \cdot \delta_E = \frac{1}{EI_{ABC}} \int_0^{L_{ABC}} M_Q\, M_P dx + \frac{1}{EI_{CDE}} \int_0^{L_{CDE}} M_Q\, M_P dx$$

$$\delta_E = \frac{-3{,}465{,}216.0 \text{ k}-\text{in}^3}{2{,}000{,}000 \text{ k}-\text{in}^2} + \frac{663{,}552 \text{ k}-\text{in}^3}{800{,}000 \text{ k}-\text{in}^2}$$

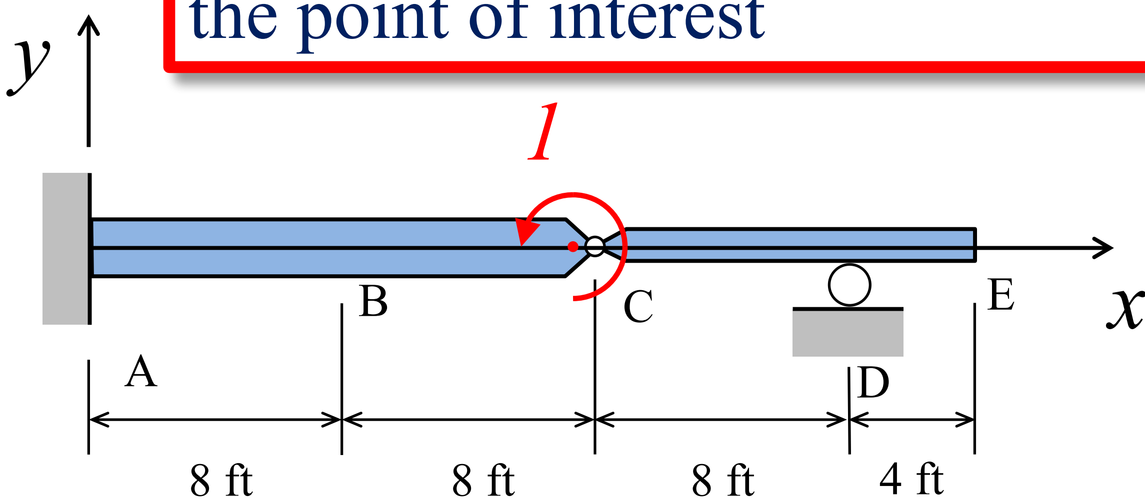
$$\delta_E = -1.733 \text{ in} + 0.8294 \text{ in} = -\,0.903 \text{ in}$$

Negative result, so deflection is in the opposite direction of the virtual unit load

$$\delta_E = 0.903 \text{ in upward}$$

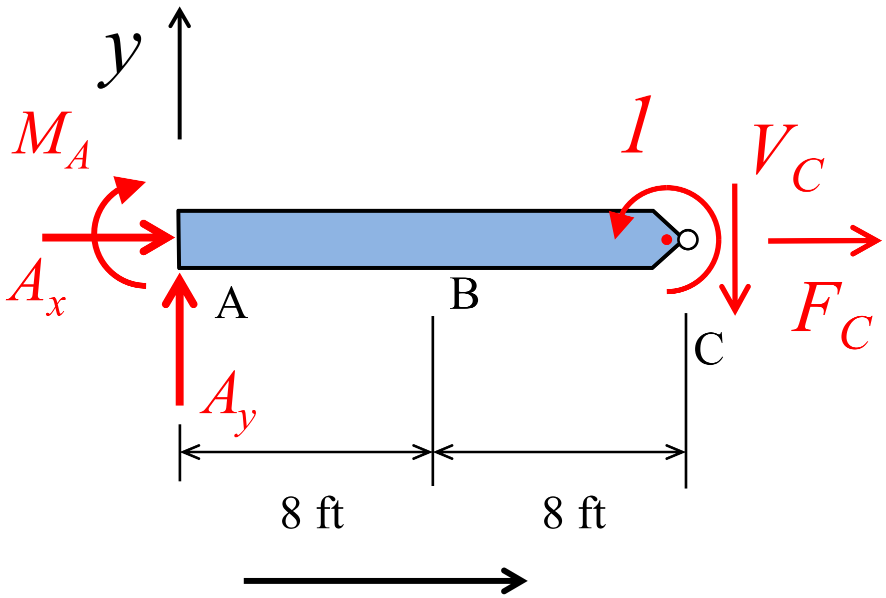
Find the Rotation Just to the Left of Point C

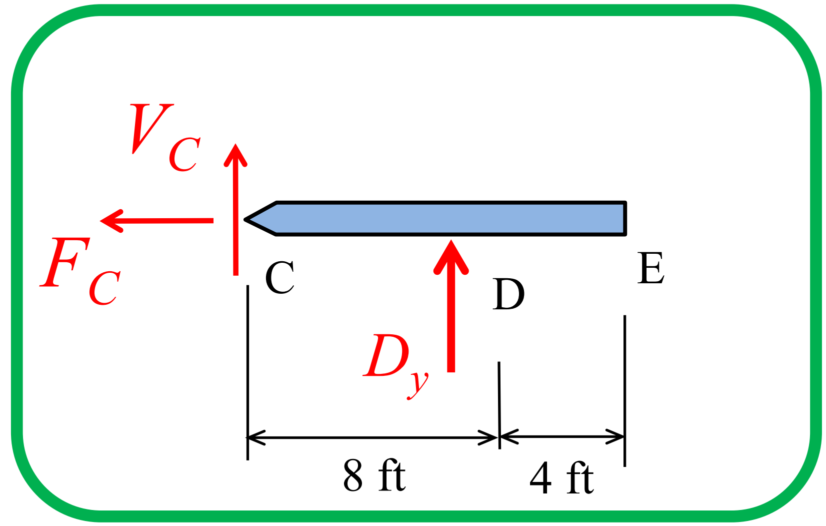
Step 1 – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest



From an equilibrium analysis, find the internal bending moment function for the virtual system: $M_Q(x)$

Find the Moment Diagram for the Virtual System





$\circlearrowleft + \sum M_A = 0 \rightarrow$ $M_A = 1$

$\xrightarrow{+} \sum F_x = 0 \rightarrow$ $A_x = 0$

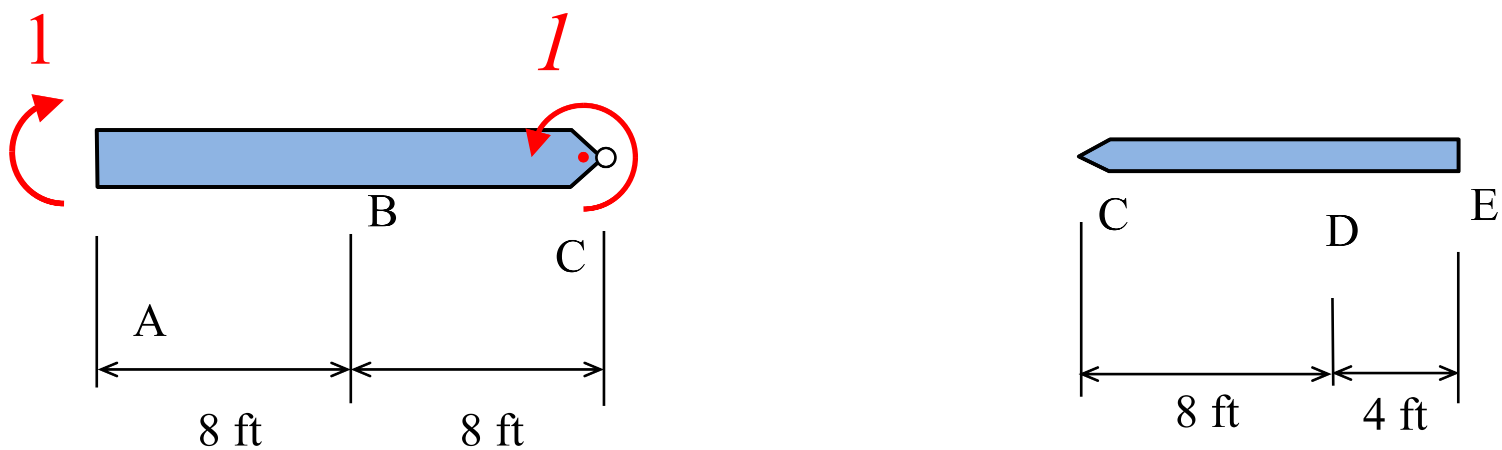
$+\uparrow \sum F_y = 0 \rightarrow$ $A_y = 0$

$\circlearrowleft + \sum M_C = 0 \rightarrow$ $D_y = 0$

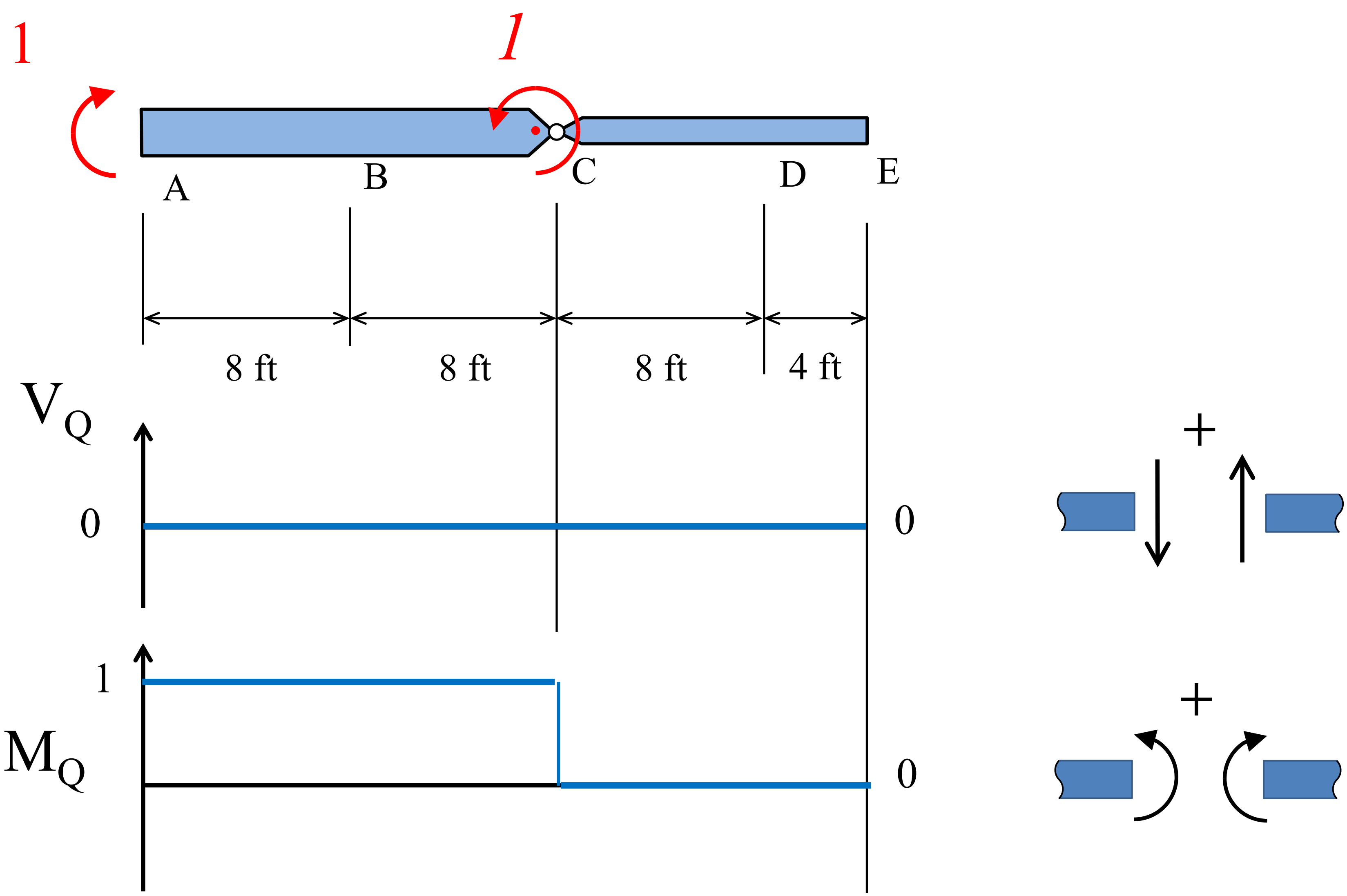
$\xrightarrow{+} \sum F_x = 0 \rightarrow$ $F_B = 0$

$+\uparrow \sum F_y = 0 \rightarrow$ $V_C = 0$

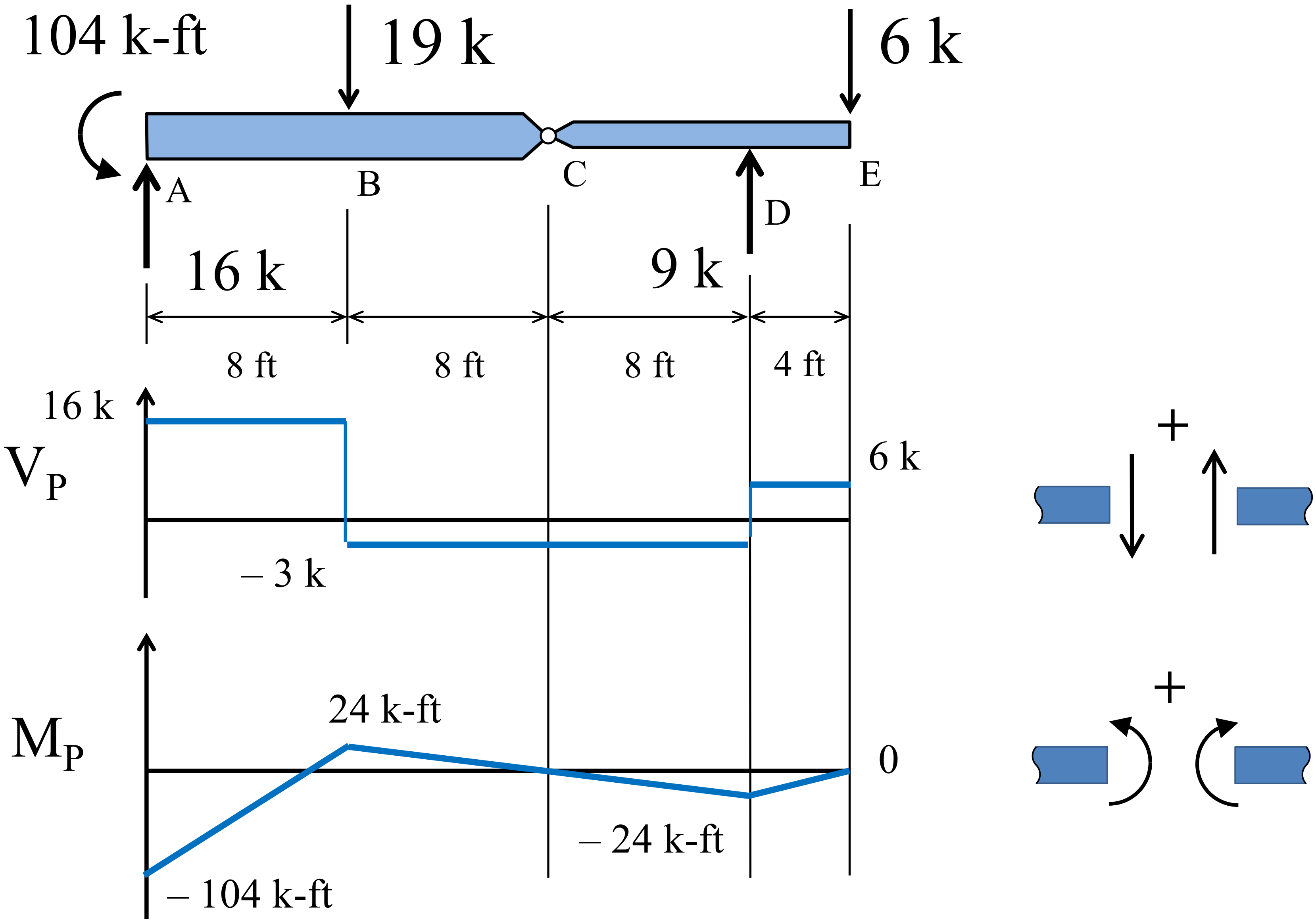
Support Reactions for the Virtual System



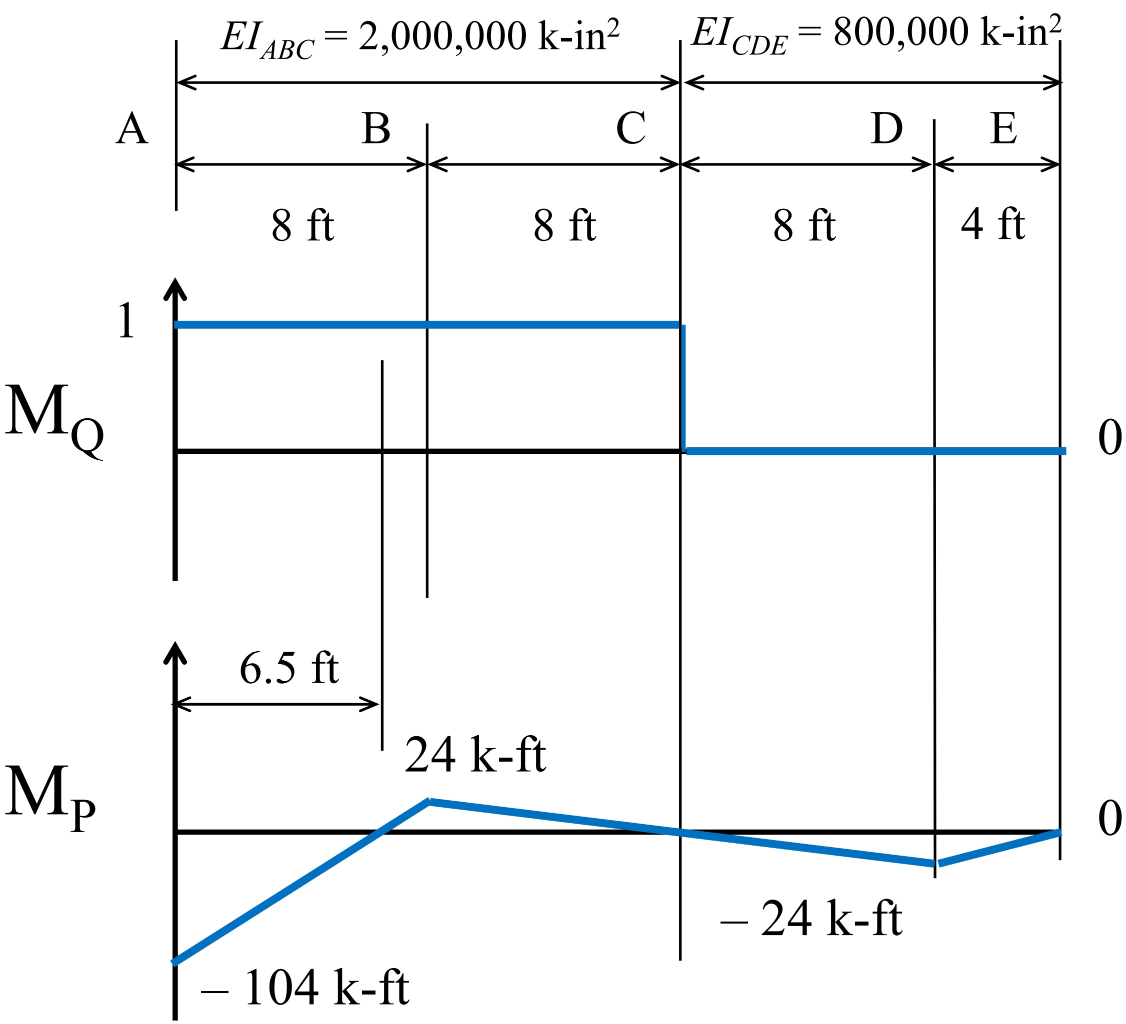
Moment Diagram for the Virtual System



Moment Diagram for the Real System



Evaluate the Virtual Work Product Integrals



$$1 \cdot \theta_{C^-} = \frac{1}{EI}\int_0^L M_Q\, M_P\, dx$$

Use Table to evaluate product integrals

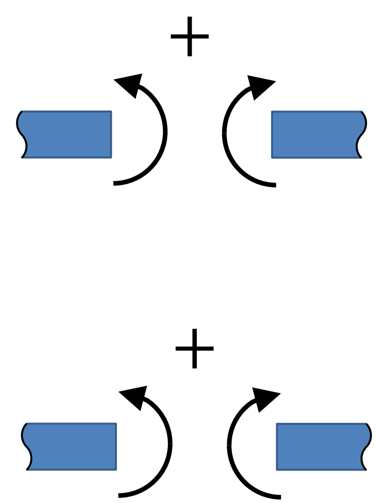


Table to Evaluate Virtual Work Product Integrals

Appendix Table.2

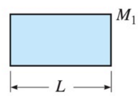
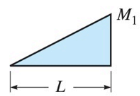
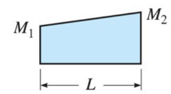
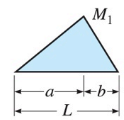
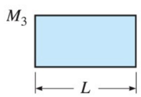
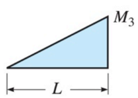
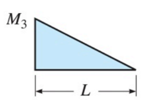
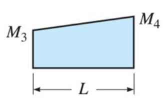
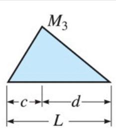
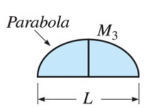
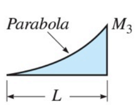
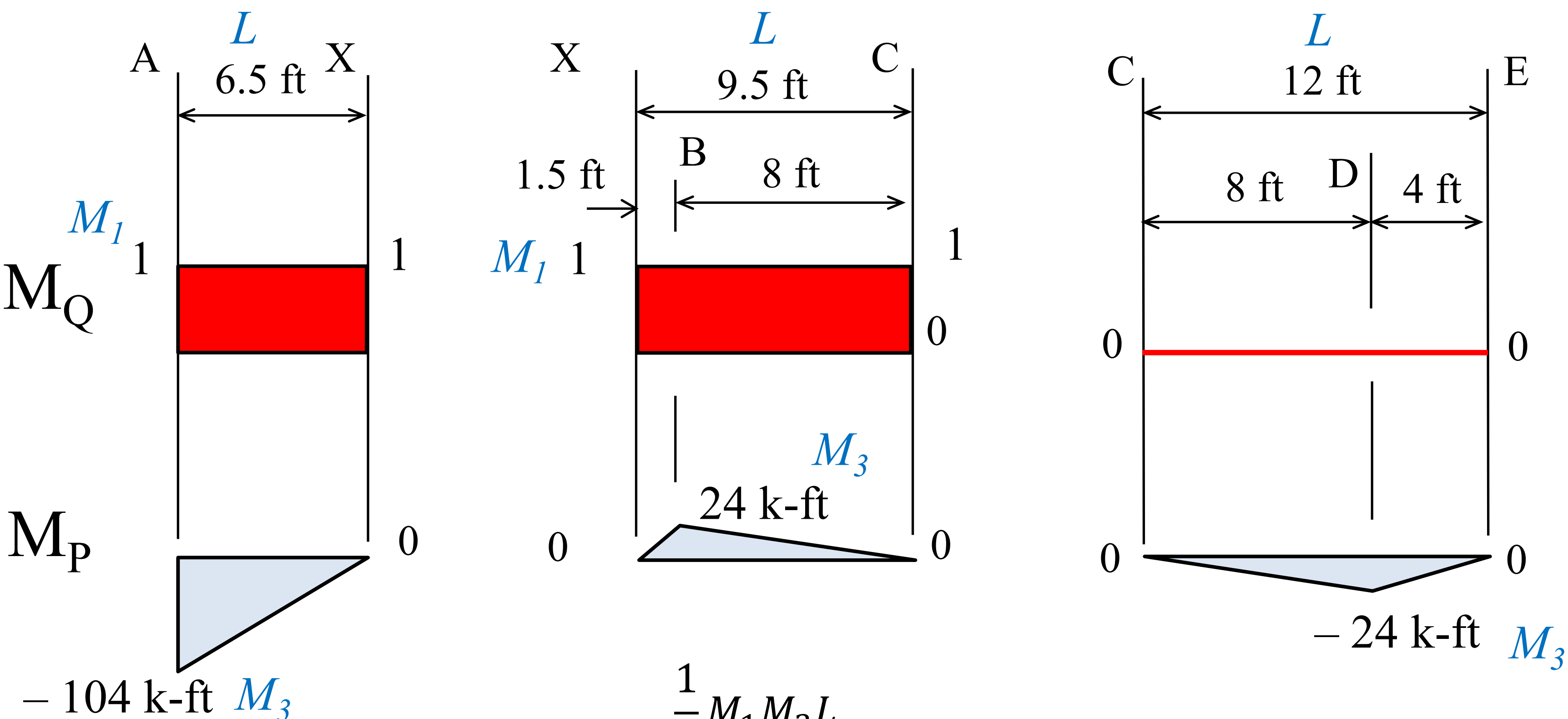
M_Q \ M_P				
	M_1M_3L	$\frac{1}{2}M_1M_3L$	$\frac{1}{2}(M_1+M_2)M_3L$	$\frac{1}{2}M_1M_3L$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{3}M_1M_3L$	$\frac{1}{6}(M_1+2M_2)M_3L$	$\frac{1}{6}M_1M_3(L+a)$
	AX $\frac{1}{2}M_1M_3L$	$\frac{1}{6}M_1M_3L$	$\frac{1}{6}(2M_1+M_2)M_3L$	$\frac{1}{6}M_1M_3(L+b)$
	$\frac{1}{2}M_1(M_3+M_4)L$	$\frac{1}{6}M_1(M_3+2M_4)L$	$\frac{1}{6}M_1(2M_3+M_4)L$ $+\frac{1}{6}M_2(M_3+2M_4)L$	$\frac{1}{6}M_1M_3(L+b)$ $+\frac{1}{6}M_1M_4(L+a)$
	XC $\frac{1}{2}M_1M_3L$	$\frac{1}{6}M_1M_3(L+c)$	$\frac{1}{6}M_1M_3(L+d)$ $+\frac{1}{6}M_2M_3(L+c)$	for $c \le a$: $\left(\frac{1}{3}-\frac{(a-c)^2}{6ad}\right)M_1M_3L$
	$\frac{2}{3}M_1M_3L$	$\frac{1}{3}M_1M_3L$	$\frac{1}{3}(M_1+M_2)M_3L$	$\frac{1}{3}M_1M_3\left(L+\frac{ab}{L}\right)$
	$\frac{1}{3}M_1M_3L$	$\frac{1}{4}M_1M_3L$	$\frac{1}{12}(M_1+3M_2)M_3L$	$\frac{1}{12}M_1M_3\left(3a+\frac{a^2}{L}\right)$

Table is as useful tool to evaluate product integrals of the form:

$$\int_0^L M_Q\, M_P\, dx$$

Evaluate Product Integrals



$$\frac{1}{2}M_1M_3L$$

$$\frac{1}{2}(1)(-104)(6.5)$$

– 338 k-ft²

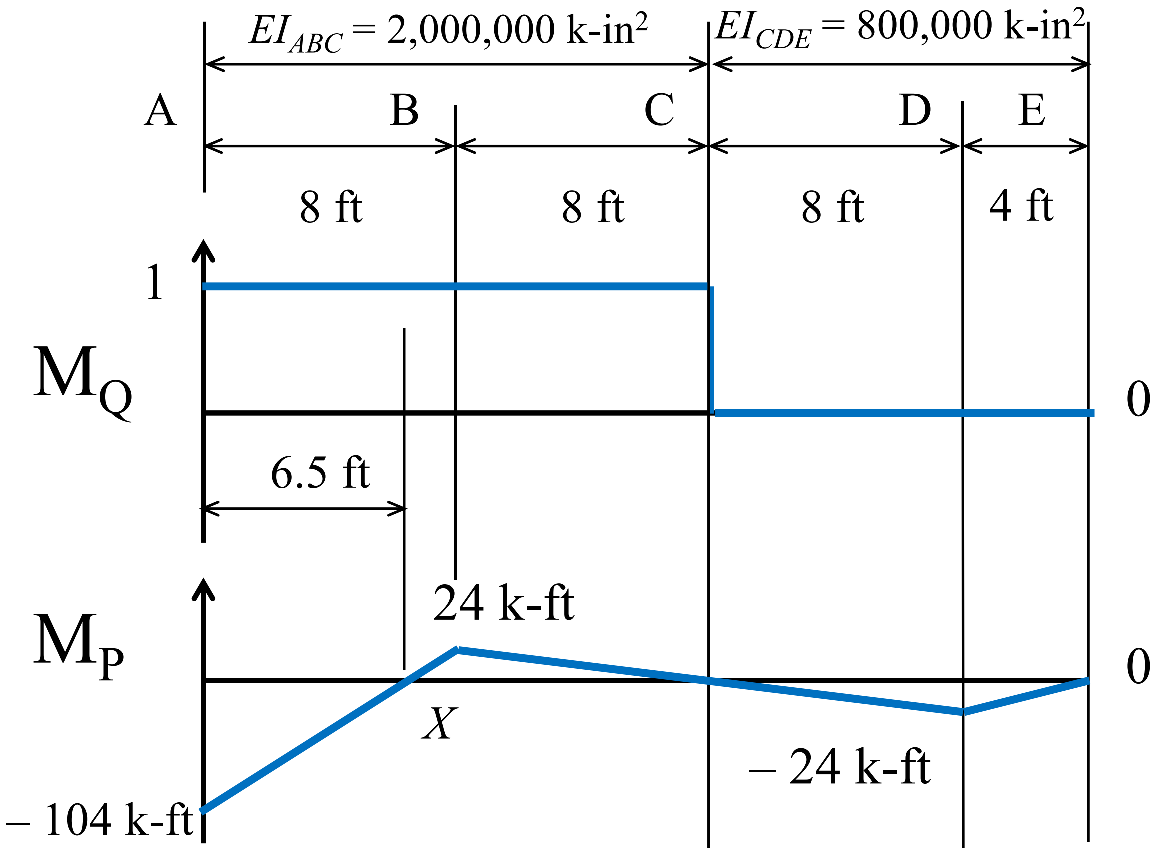
$$\frac{1}{2}M_1M_3L$$

$$\frac{1}{2}(1)(24)(9.5)$$

114 k-ft²

0

Evaluate Product Integrals



$$1 \cdot \theta_{C^-} = \frac{1}{EI}\int_0^L M_Q\, M_P\, dx$$

Segment AX

– 338 k-ft^2

Segment XC

114 k-ft^2

Segment CDE

0

$$\int_0^{L_{ABC}} M_Q\, M_P\, dx = \left(-338 + 114\ \text{k-ft}^2\right)\left(\frac{12^2\ \text{in}^2}{\text{ft}^2}\right) = -32{,}256\ \text{k-in}^2$$

$$\int_0^{L_{CDE}} M_Q\, M_P\, dx = 0$$

Evaluate Product Integrals

$$\int_0^{L_{ABC}} M_Q\, M_P dx = (-338 + 114\ \text{k–ft}^2)\left(\frac{12^2\ \text{in}^2}{\text{ft}^2}\right) = -32{,}256\ \text{k–in}^2$$

$$\int_0^{L_{CDE}} M_Q\, M_P dx = 0$$

$$1 \cdot \theta_{C^-} = \frac{1}{EI_{ABC}} \int_0^{L_{ABC}} M_Q\, M_P dx + \frac{1}{EI_{CDE}} \int_0^{L_{CDE}} M_Q\, M_P dx$$

$$\theta_{C^-} = \frac{-32{,}256\ \ \text{k–in}^2}{2{,}000{,}000\ \text{k–in}^2} + \frac{0}{800{,}000\ \text{k–in}^2}$$

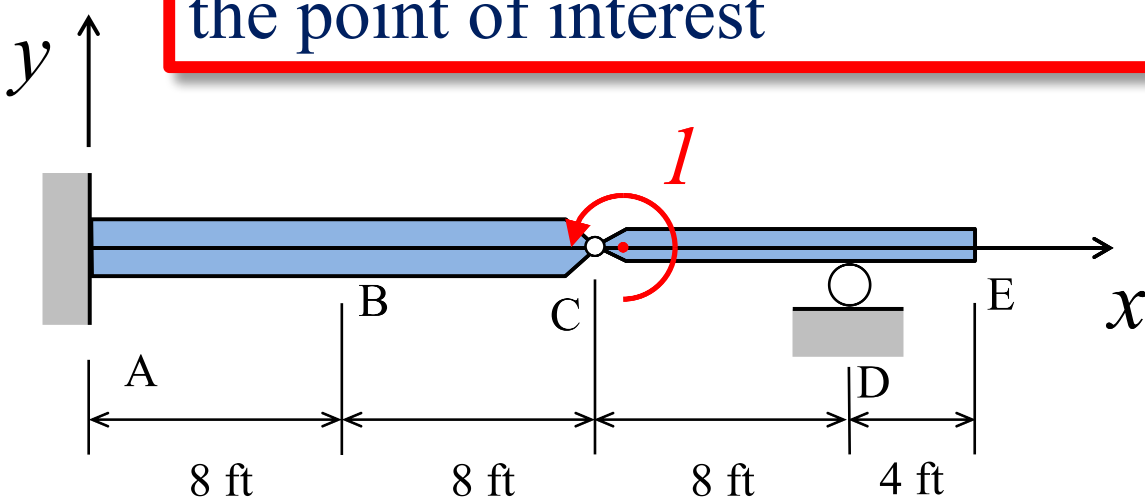
$$\theta_{C^-} = -0.0161\ \text{rad} + 0 = -\ 0.0161\ \text{rad}$$

Negative result, so rotation is in the opposite direction of the virtual unit moment

$$\theta_{C^-} = 0.0161\ \text{radians clockwise}$$

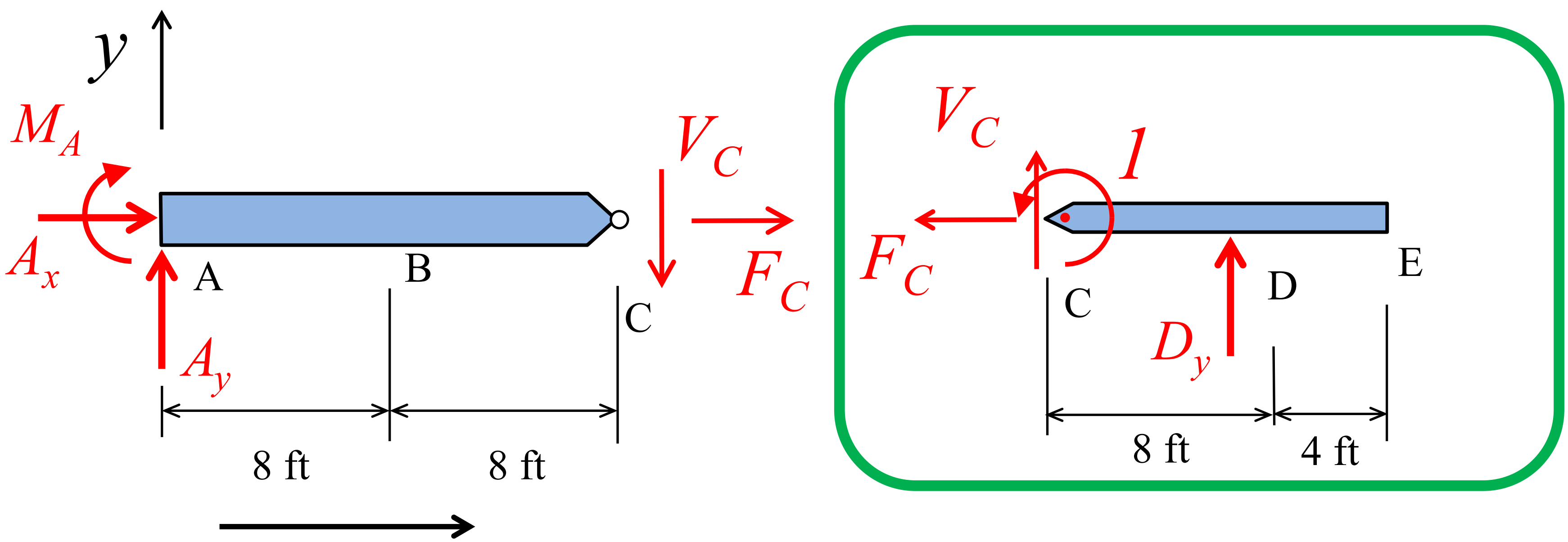
Find the Rotation Just to the Right of Point C

Step 1 – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest



From an equilibrium analysis, find the internal bending moment function for the virtual system: $M_Q(x)$

Find the Moment Diagram for the Virtual System



$\circlearrowleft + \sum M_A = 0 \rightarrow$ $M_A = -2$

$\overset{+}{\rightarrow} \sum F_x = 0 \rightarrow$ $A_x = 0$

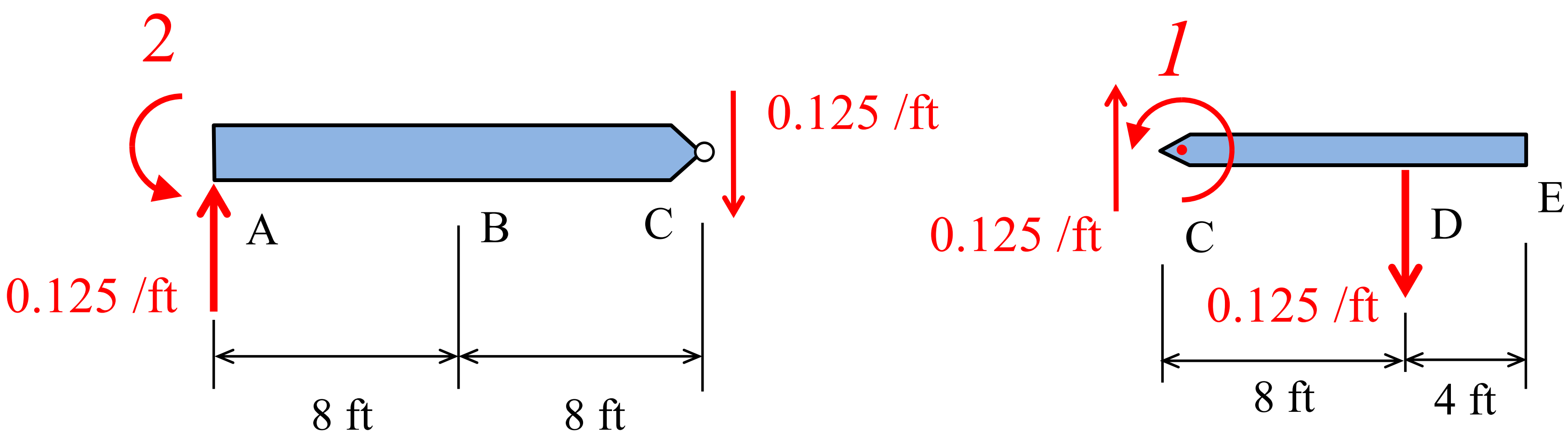
$+\uparrow \sum F_y = 0 \rightarrow$ $A_y = 0.125\ /\text{ft}$

$\circlearrowleft + \sum M_C = 0 \rightarrow$ $D_y = -0.125\ /\text{ft}$

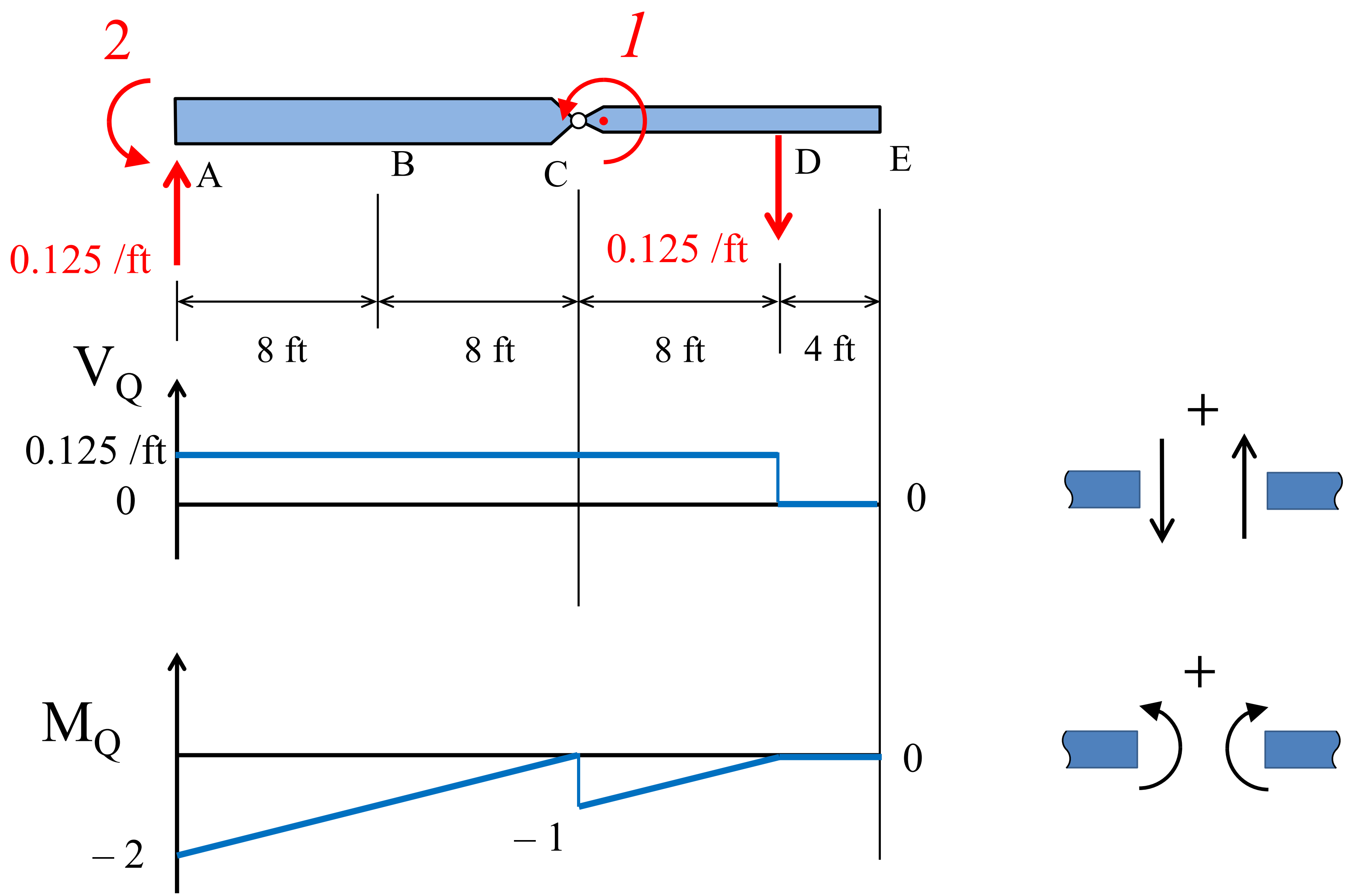
$\overset{+}{\rightarrow} \sum F_x = 0 \rightarrow$ $F_B = 0$

$+\uparrow \sum F_y = 0 \rightarrow$ $V_C = 0.125\ /\text{ft}$

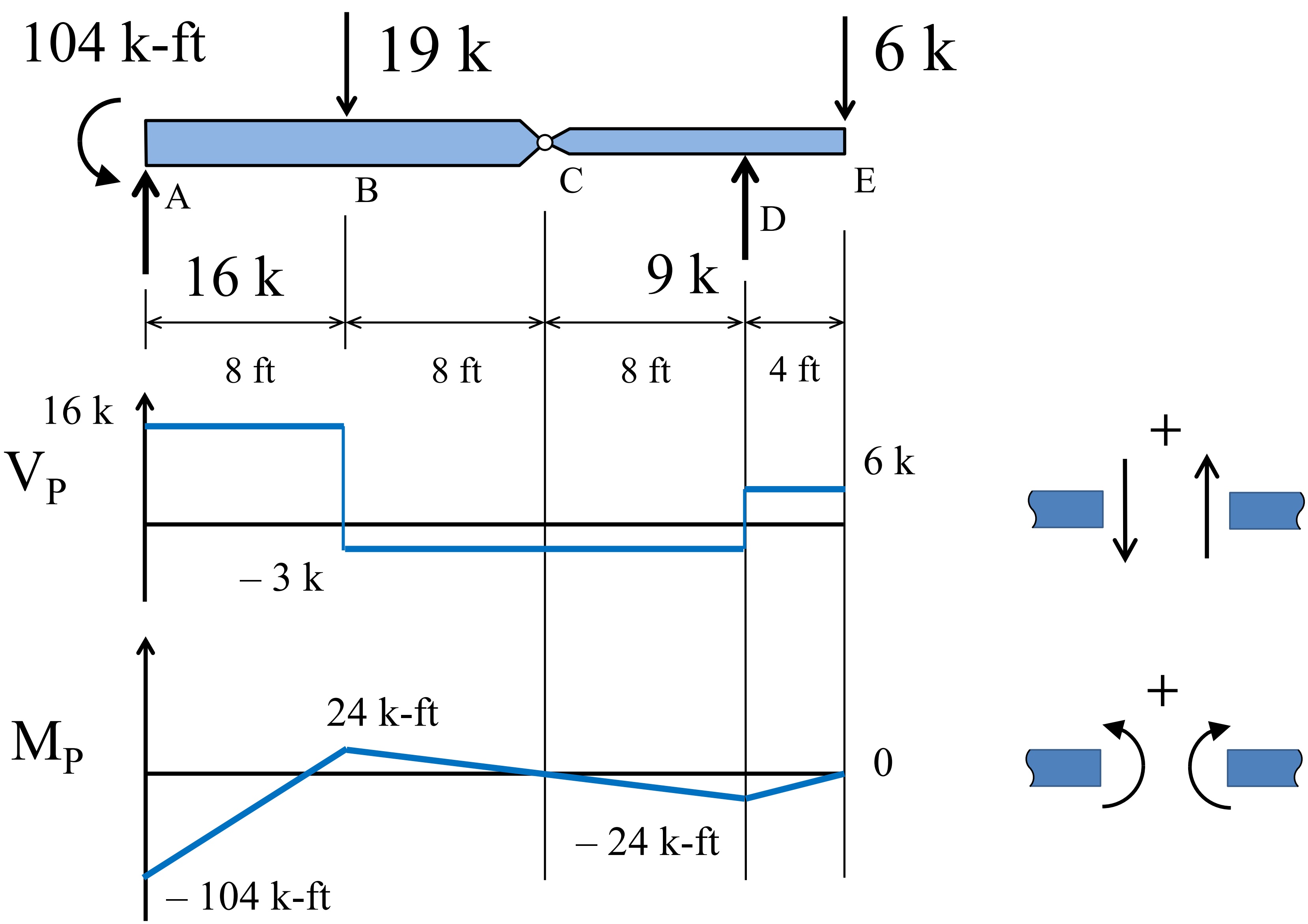
Support Reactions for the Virtual System



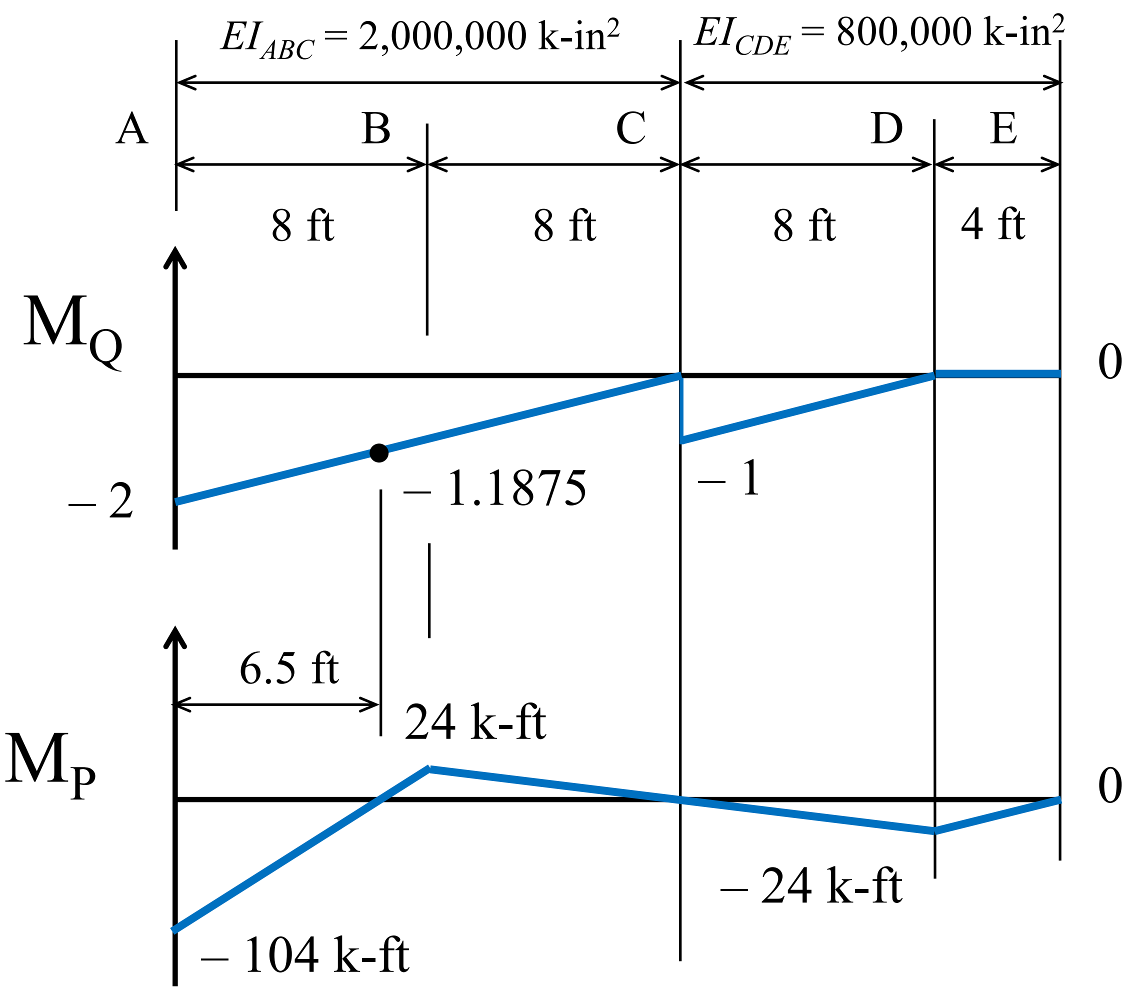
Moment Diagram for the Virtual System



Moment Diagram for the Real System



Evaluate the Virtual Work Product Integrals



$$1 \cdot \theta_{C^+} = \frac{1}{EI} \int_0^L M_Q \, M_P \, dx$$

Use Table to evaluate product integrals

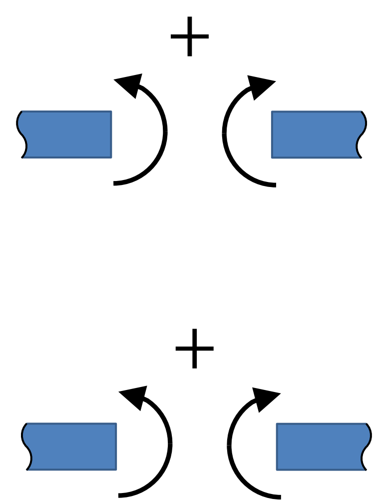


Table to Evaluate Virtual Work Product Integrals

Appendix Table.2

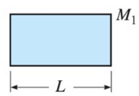
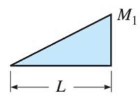
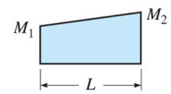
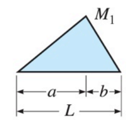
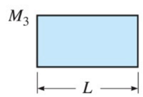
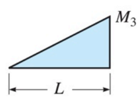
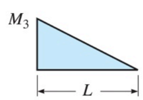
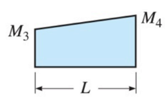
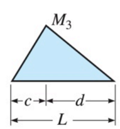
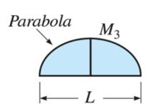
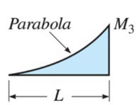
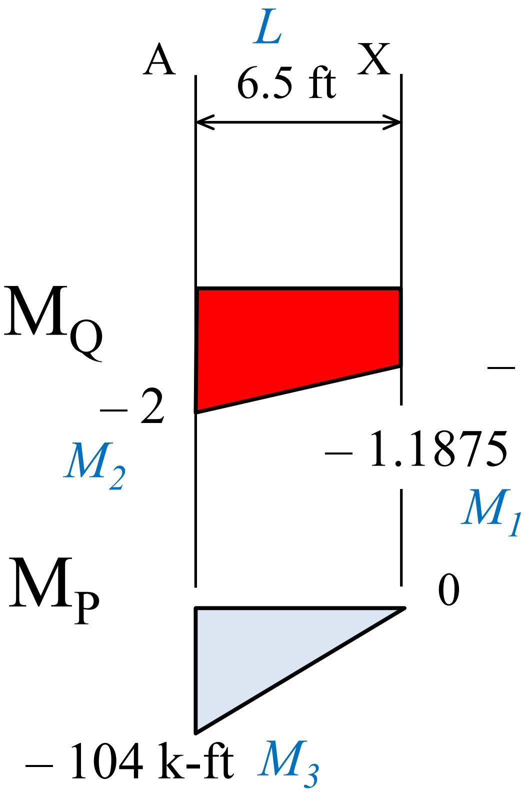
M_P \ M_Q				
	M_1M_3L	$\frac{1}{2}M_1M_3L$	$\frac{1}{2}(M_1+M_2)M_3L$	$\frac{1}{2}M_1M_3L$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{3}M_1M_3L$	AX $\frac{1}{6}(M_1+2M_2)M_3L$	$\frac{1}{6}M_1M_3(L+a)$
	$\frac{1}{2}M_1M_3L$	CD $\frac{1}{6}M_1M_3L$	$\frac{1}{6}(2M_1+M_2)M_3L$	$\frac{1}{6}M_1M_3(L+b)$
	$\frac{1}{2}M_1(M_3+M_4)L$	$\frac{1}{6}M_1(M_3+2M_4)L$	$\frac{1}{6}M_1(2M_3+M_4)L$ $+\frac{1}{6}M_2(M_3+2M_4)L$	$\frac{1}{6}M_1M_3(L+b)$ $+\frac{1}{6}M_1M_4(L+a)$
	$\frac{1}{2}M_1M_3L$	XC $\frac{1}{6}M_1M_3(L+c)$	$\frac{1}{6}M_1M_3(L+d)$ $+\frac{1}{6}M_2M_3(L+c)$	for $c \le a$: $\left(\frac{1}{3}-\frac{(a-c)^2}{6ad}\right)M_1M_3L$
	$\frac{2}{3}M_1M_3L$	$\frac{1}{3}M_1M_3L$	$\frac{1}{3}(M_1+M_2)M_3L$	$\frac{1}{3}M_1M_3\left(L+\frac{ab}{L}\right)$
	$\frac{1}{3}M_1M_3L$	$\frac{1}{4}M_1M_3L$	$\frac{1}{12}(M_1+3M_2)M_3L$	$\frac{1}{12}M_1M_3\left(3a+\frac{a^2}{L}\right)$

Table is as useful tool to evaluate product integrals of the form:

$$\int_0^L M_Q\, M_P\, dx$$

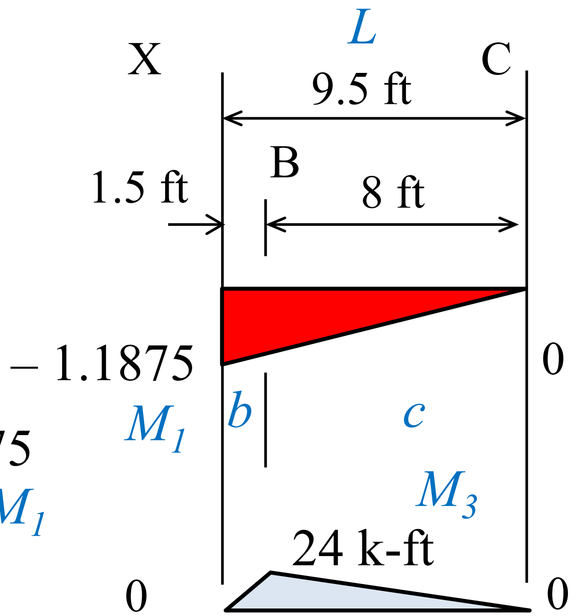
Evaluate Product Integrals Using the Table



$$\frac{1}{6}(M_1 + 2M_2)M_3L$$

$$\frac{1}{6}\big(-1.1875 + 2(-2)\big)(-104)(6.5)$$

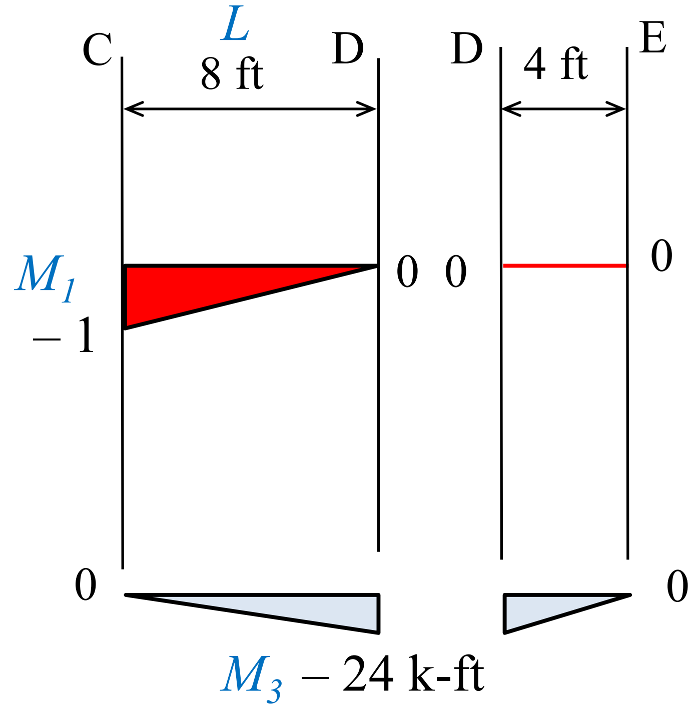
584.458 k-ft²



$$\frac{1}{6}M_1M_3(L + c)$$

$$\frac{1}{6}(-1.1875)(24)(9.5 + 8)$$

– 83.125 k-ft²



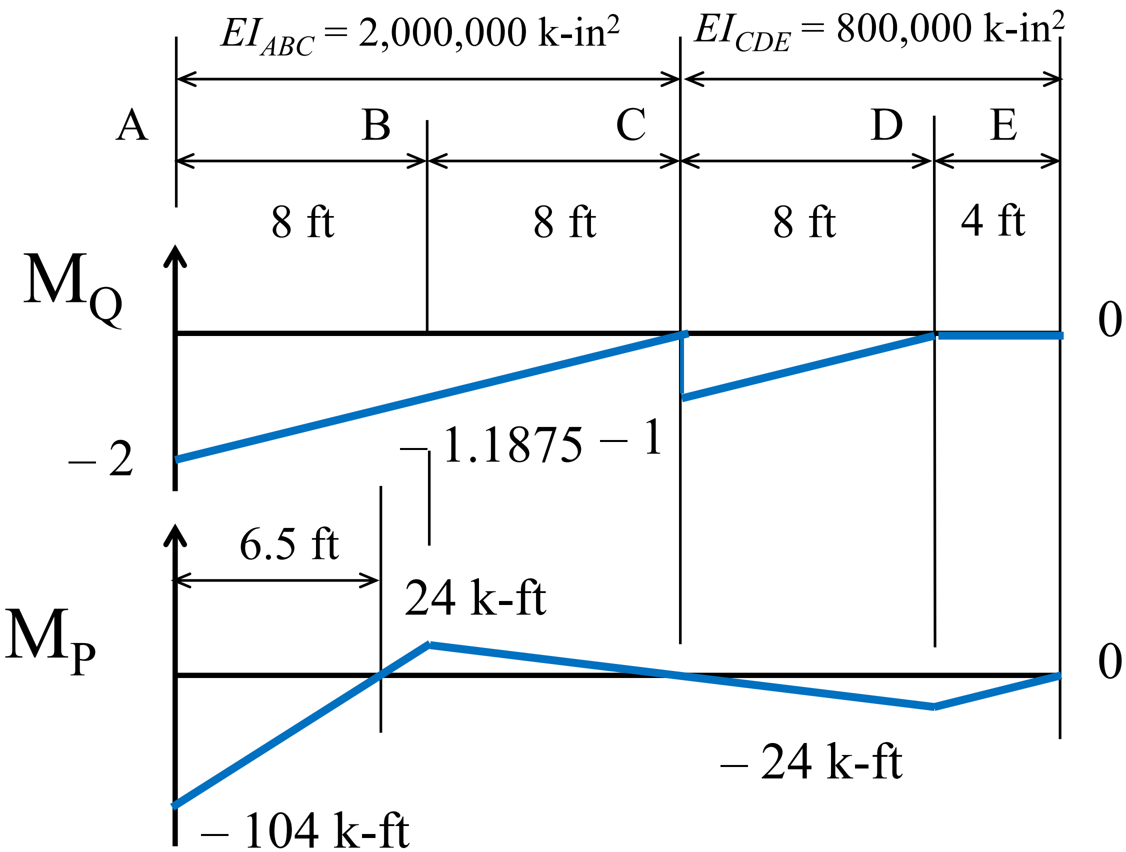
$$\frac{1}{6}M_1M_3L$$

$$\frac{1}{6}(-1)(-24)(8)$$

32 k-ft²

0

Evaluate the Virtual Work Product Integrals



$$1 \cdot \theta_{C^+} = \frac{1}{EI} \int_0^L M_Q \, M_P \, dx$$

Segment AX

584.458 k-ft^2

Segment XC

– 83.125 k-ft^2

Segment CD

32 k-ft^2

Segment DE

0

$$\int_0^{L_{ABC}} M_Q \, M_P \, dx = \left(584.458 - 83.125 \text{ k–ft}^2\right)\left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right) = 72{,}191.95 \text{ k–in}^2$$

$$\int_0^{L_{CDE}} M_Q \, M_P \, dx = \left(32 \text{ k–ft}^2\right)\left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right) = 4608 \text{ k–in}^2$$

Evaluate Product Integrals

$$\int_0^{L_{ABC}} M_Q\, M_P dx = \left(584.458 - 83.125 \text{ k−ft}^2\right)\left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right) = 72{,}191.95 \text{ k−in}^2$$

$$\int_0^{L_{CDE}} M_Q\, M_P dx = \left(32 \text{ k−ft}^2\right)\left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right) = 4608 \text{ k−in}^2$$

$$1 \cdot \theta_{C^+} = \frac{1}{EI_{ABC}}\int_0^{L_{ABC}} M_Q\, M_P dx + \frac{1}{EI_{CDE}}\int_0^{L_{CDE}} M_Q\, M_P dx$$

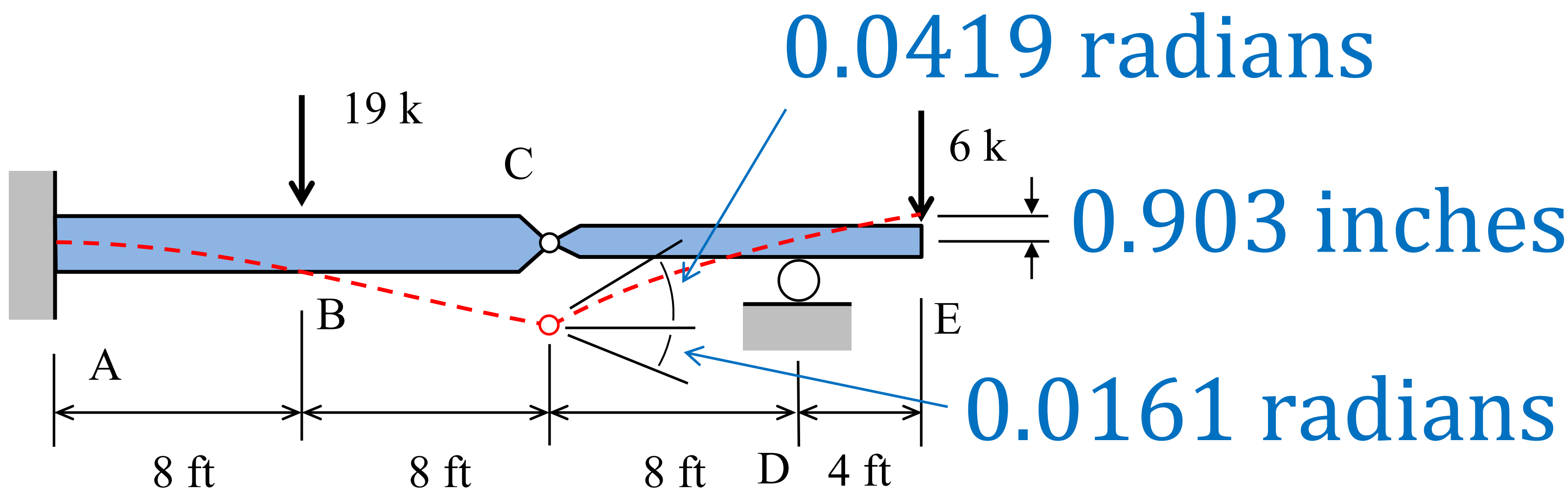
$$\theta_{C^+} = \frac{72{,}191.95 \text{ k−in}^2}{2{,}000{,}000 \text{ k−in}^2} + \frac{4608 \text{ k−in}^2}{800{,}000 \text{ k−in}^2}$$

$$\theta_{C^+} = 0.0361 + 0.00576 \text{ rad} = 0.0419 \text{ rad}$$

Positive result, so rotation is in the same direction of the virtual unit moment

$$\theta_{C^+} = 0.0419 \text{ radians counter−clockwise}$$

Beam Deflection Example Results



The overhanging beam shown has a fixed support at A, a roller support at C and an internal hinge at B. EI_{ABC} = 2,000,000 k-in^2 and EI_{CDE} = 800,000 k-in^2

For the loads shown, find the following:

1. The vertical deflection at point E;
2. The slope just to the left of the internal hinge at C;
3. The slope just to the right of the internal hinge at C