Moment of a Force About a Point in Three-Dimensions

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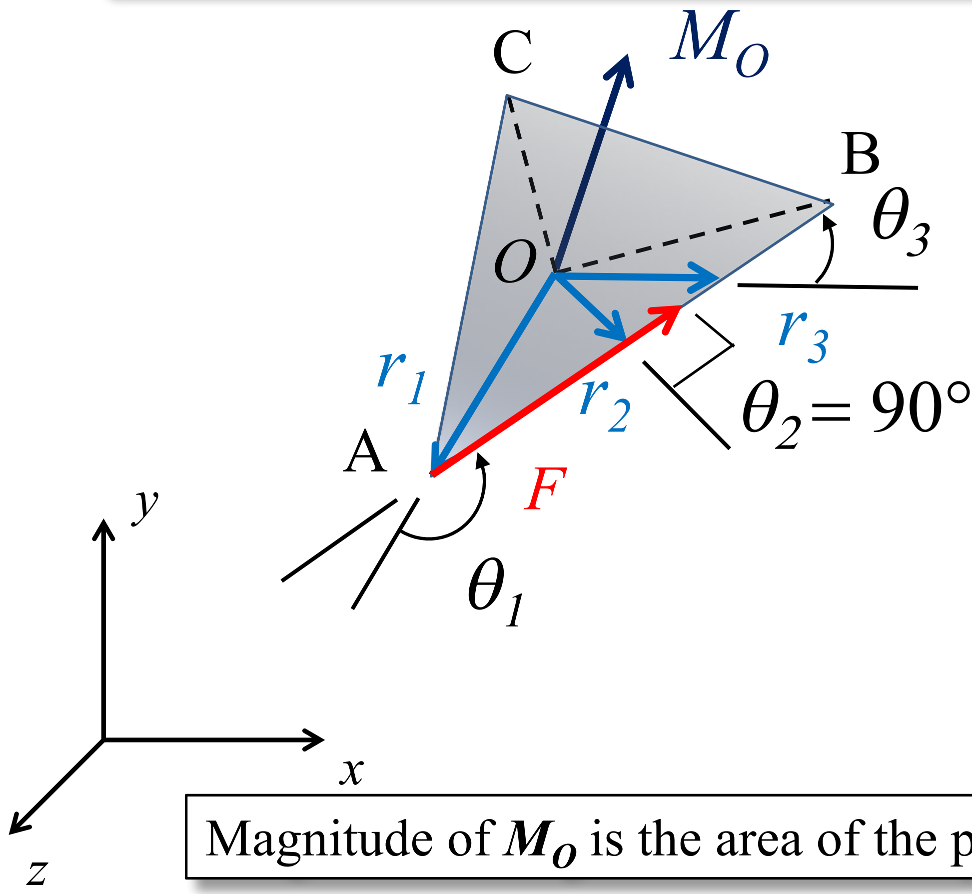
Recall the Definition of the Moment of a Force *F* about a Point *O*



$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

$$M_O = rF \sin \theta$$

r is a position vector that must satisfy:

- Tail of ***r*** is at point ***O***;
- Tip can be on any point on the line-of-action of ***F***

Magnitude of $\boldsymbol{M_O}$ is the area of the parallelogram defined by ***r*** and ***F***

Direction of $\boldsymbol{M_O}$ is perpendicular to the plane defined by ***r*** and ***F***

Sense of $\boldsymbol{M_O}$ is defined by the right-hand rule

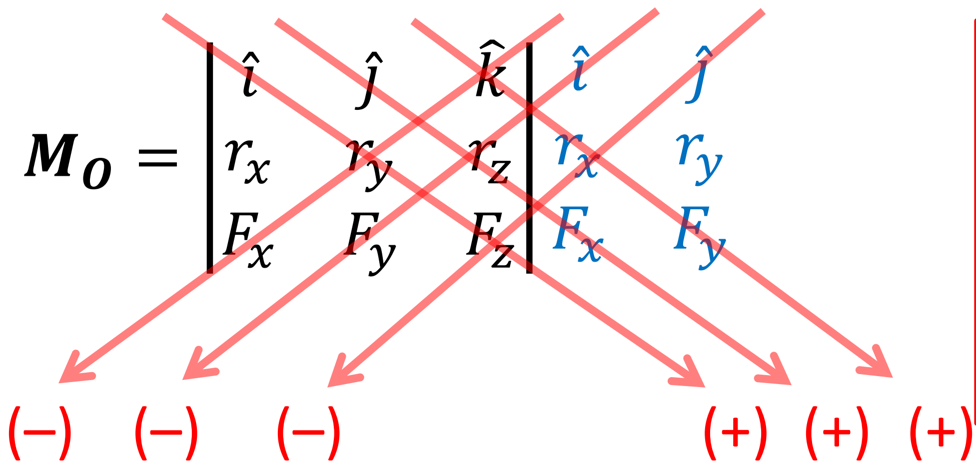
Moment of a Force about a Point when the Position Vector and Force Vector are in Cartesian Vector Form

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

$$\boldsymbol{r} = r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}$$

$$\boldsymbol{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

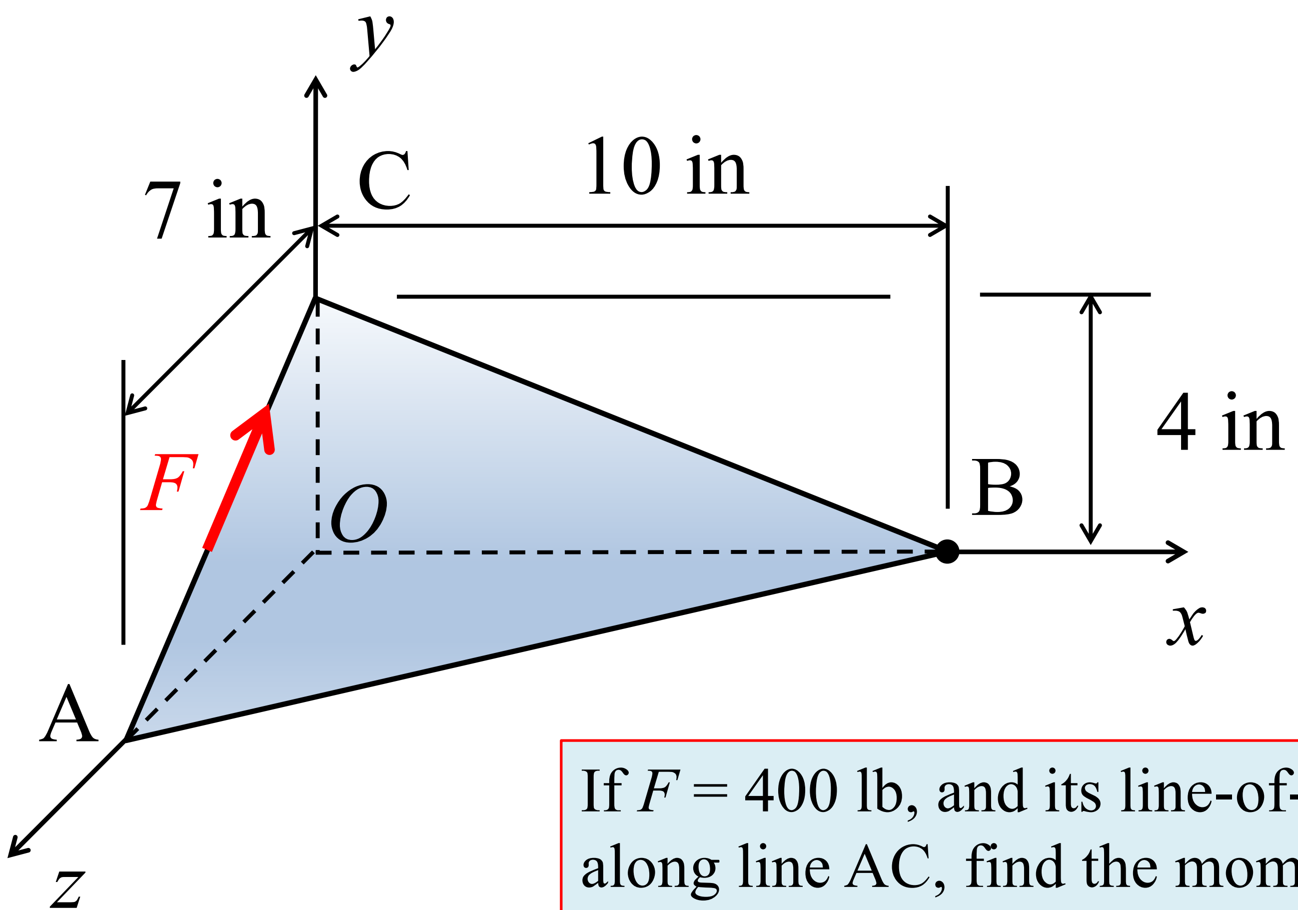
$$\boldsymbol{M_O} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$



For the moment of a general three-dimensional force about a point, it is almost always easiest to express the force and position vectors in Cartesian vector form

$$\boldsymbol{M_O} = (r_yF_z - r_zF_y)\hat{\imath} + (r_zF_x - r_xF_z)\hat{\jmath} + (r_xF_y - r_yF_x)\hat{k}$$

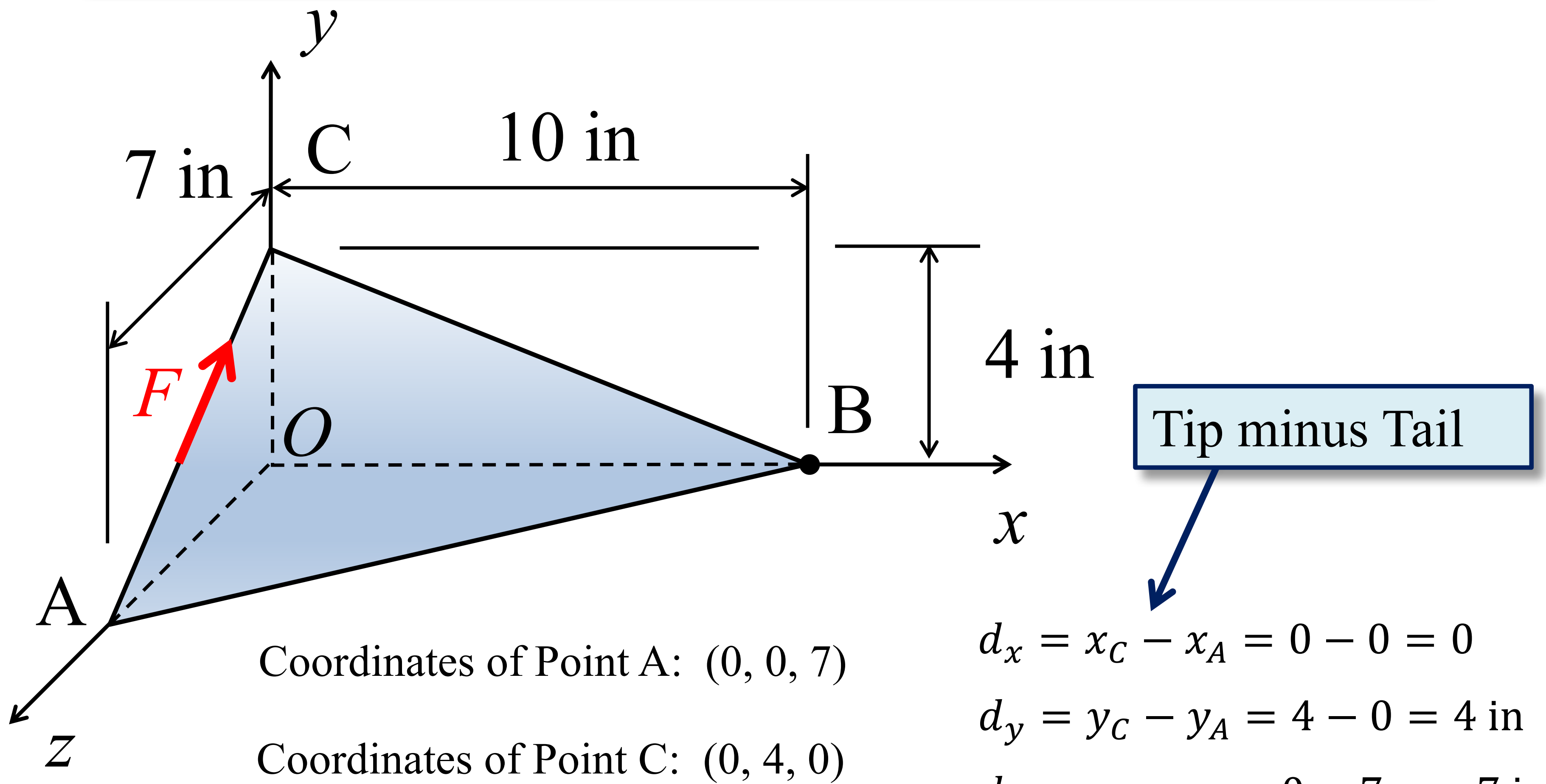
Example Problem



If $F = 400$ lb, and its line-of-action lies along line AC, find the moment of the force about point B using:

1. Position vector AB;
2. Position vector CB.

Express *F* in Cartesian Vector Form



Coordinates of Point A: (0, 0, 7)

Coordinates of Point C: (0, 4, 0)

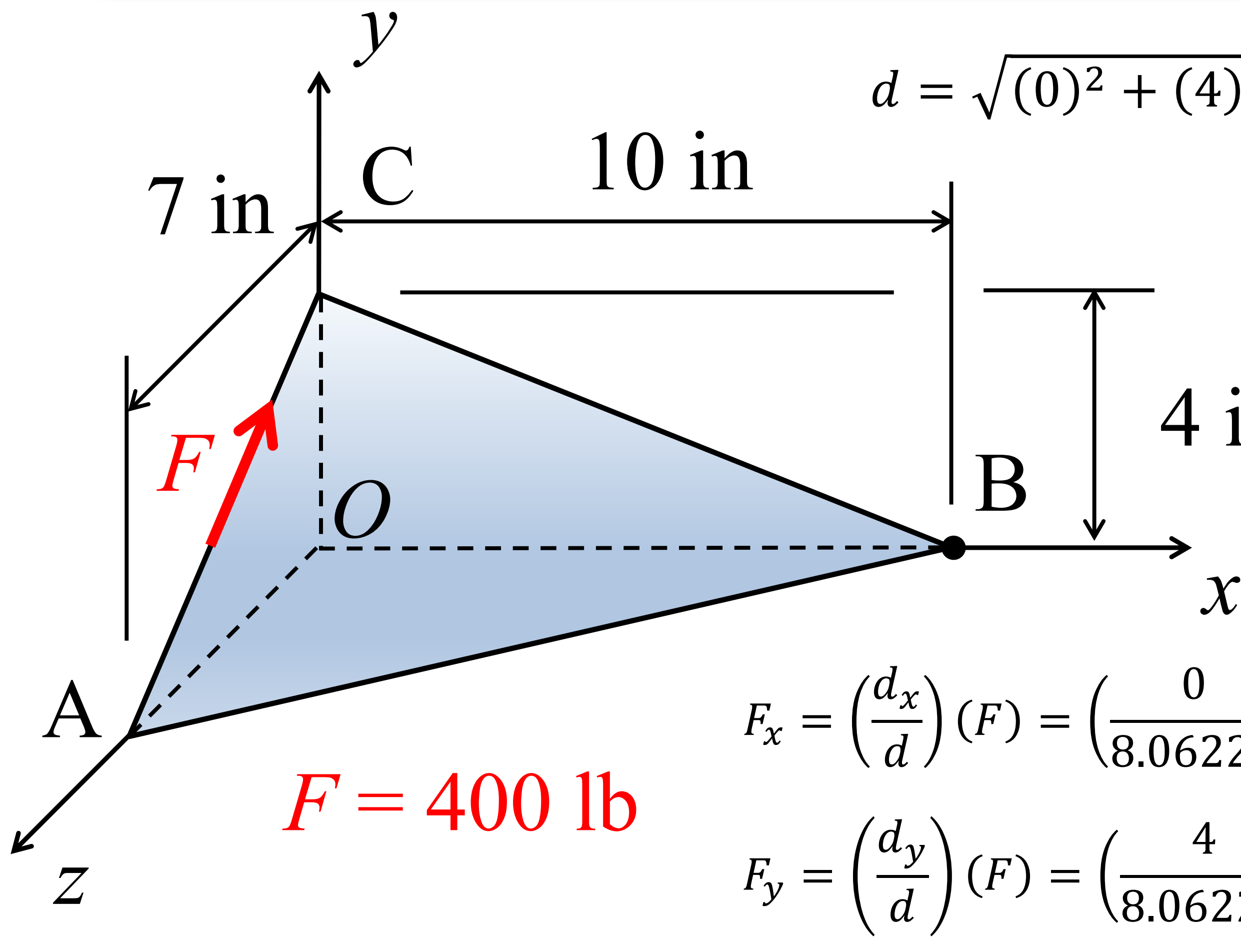
$$d_x = x_C - x_A = 0 - 0 = 0$$

$$d_y = y_C - y_A = 4 - 0 = 4 \text{ in}$$

$$d_z = z_C - z_A = 0 - 7 = -7 \text{ in}$$

$$d = \sqrt{(0)^2 + (4)^2 + (-7)^2} = 8.06226 \text{ in}$$

Express *F* in Cartesian Vector Form



$$d = \sqrt{(0)^2 + (4)^2 + (-7)^2} = 8.06226 \text{ in}$$

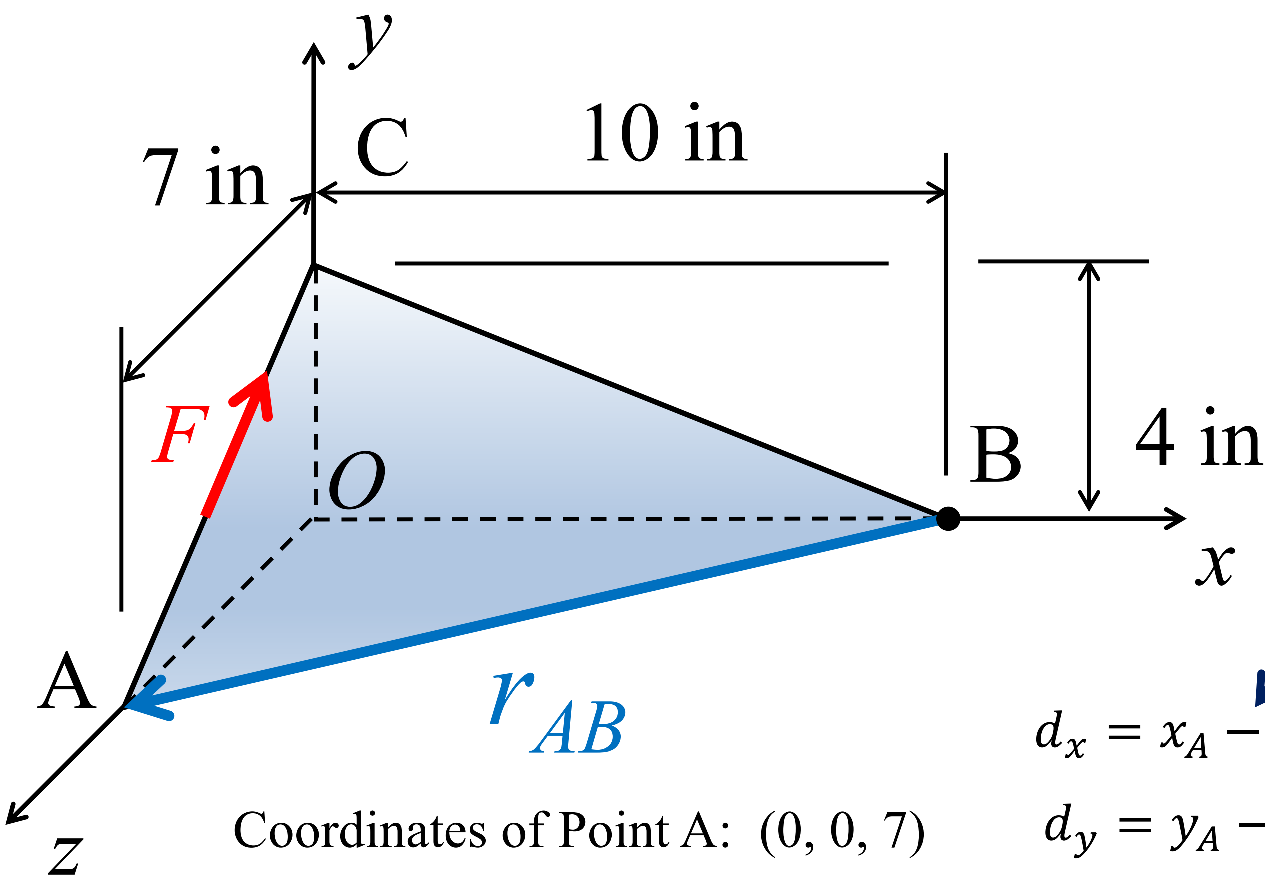
$$F_x = \left(\frac{d_x}{d}\right)(F) = \left(\frac{0}{8.06226}\right)(400 \text{ lb}) = 0$$

$$F_y = \left(\frac{d_y}{d}\right)(F) = \left(\frac{4}{8.06226}\right)(400 \text{ lb}) = 198.456 \text{ lb}$$

$$F_z = \left(\frac{d_z}{d}\right)(F) = \left(\frac{-7}{8.06226}\right)(400 \text{ lb}) = -347.30 \text{ lb}$$

$$\boldsymbol{F} = 198.456\hat{j} - 347.30\hat{k} \text{ lb}$$

Express r_{AB} in Cartesian Vector Form



Tip minus Tail

Coordinates of Point A: (0, 0, 7)

Coordinates of Point B: (10, 0, 0)

$$d_x = x_A - x_B = 0 - 10 = -10 \text{ in}$$

$$d_y = y_A - y_B = 0 - 0 = 0$$

$$d_z = z_A - z_B = 7 - 0 = 7 \text{ in}$$

$$\boldsymbol{r_{AB}} = -10\hat{\imath} + 7\hat{k} \text{ in}$$

Calculate the Moment of a Force about Point B

$$\boldsymbol{M_B} = \boldsymbol{r_{AB}} \times \boldsymbol{F}$$

$$\boldsymbol{r_{AB}} = -10\hat{\imath} + 7\hat{k} \text{ in}$$

$$\boldsymbol{F} = 198.456\hat{\jmath} - 347.30\hat{k} \text{ lb}$$

$$\boldsymbol{M_B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_{ABx} & r_{ABy} & r_{ABz} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\boldsymbol{M_B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -10 & 0 & 7 \\ 0 & 198.456 & -347.30 \end{vmatrix} \begin{matrix} \hat{\imath} & \hat{\jmath} \\ -10 & 0 \\ 0 & 198.456 \end{matrix}$$

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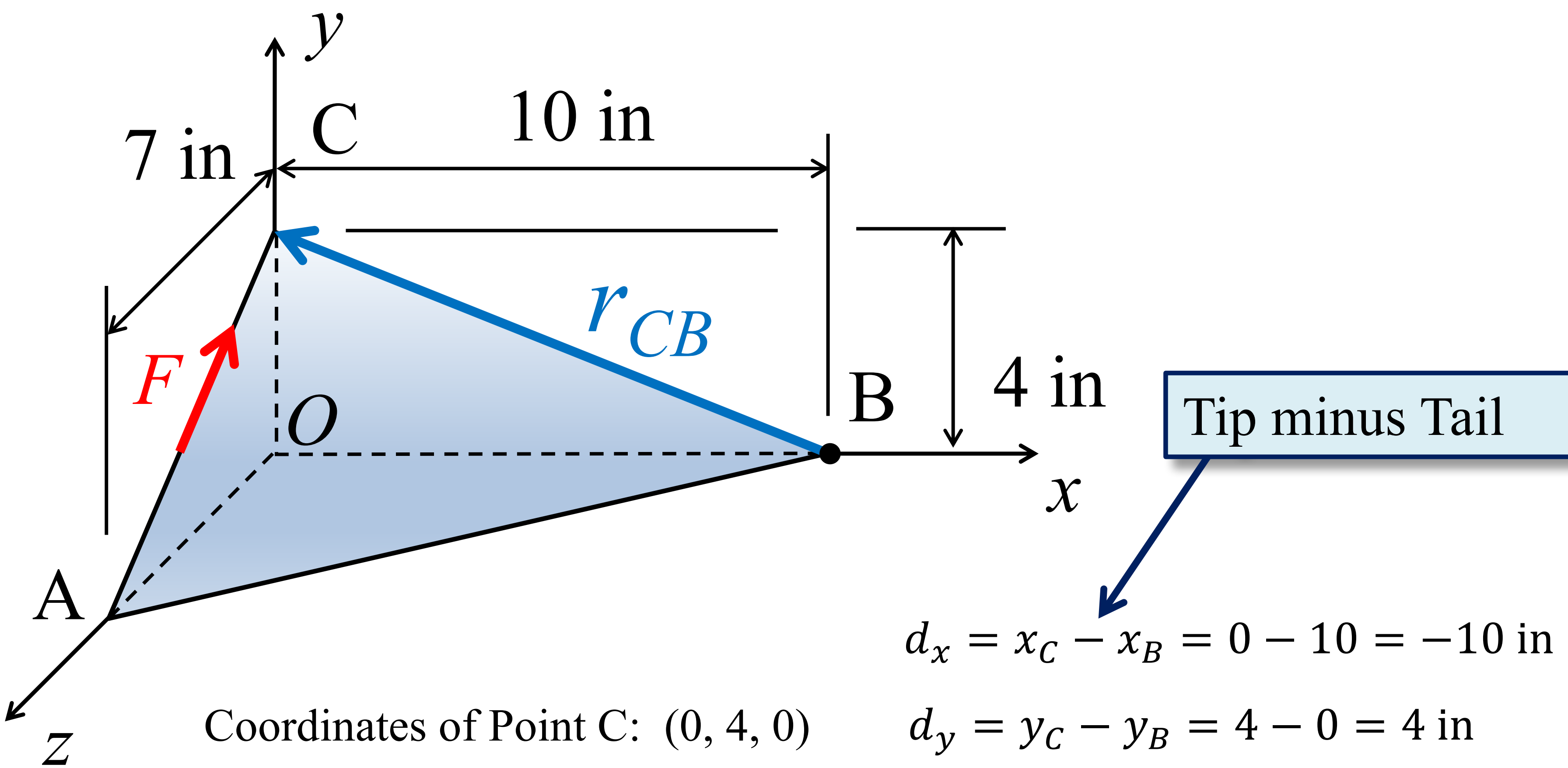
$$\boldsymbol{M_B} = (-10)(198.456)\hat{k} - (-10)(-347.30)\hat{\jmath} - (7)(198.456)\hat{\imath} \text{ lb−in}$$

$$\boldsymbol{M_B} = -1389.2\hat{\imath} - 3473.0\hat{\jmath} - 1984.6\hat{k} \text{ lb−in}$$

Express r_{CB} in Cartesian Vector Form



Coordinates of Point C: (0, 4, 0)

Coordinates of Point B: (10, 0, 0)

$$d_x = x_C - x_B = 0 - 10 = -10 \text{ in}$$

$$d_y = y_C - y_B = 4 - 0 = 4 \text{ in}$$

$$d_z = z_C - z_B = 0 - 0 = 0$$

$$\boldsymbol{r_{CB}} = -10\hat{\imath} + 4\hat{\jmath} \text{ in}$$

Calculate the Moment of a Force about Point B

$$\boldsymbol{M_B} = \boldsymbol{r_{CB}} \times \boldsymbol{F}$$

$$\boldsymbol{r_{CB}} = -10\hat{\imath} + 4\hat{\jmath} \ \text{in}$$

$$\boldsymbol{F} = 198.456\hat{\jmath} - 347.30\hat{k} \ \text{lb}$$

$$\boldsymbol{M_B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_{CBx} & r_{CBy} & r_{CBz} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\boldsymbol{M_B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -10 & 4 & 0 \\ 0 & 198.456 & -347.30 \end{vmatrix} \begin{matrix} \hat{\imath} & \hat{\jmath} \\ -10 & 4 \\ 0 & 198.456 \end{matrix}$$

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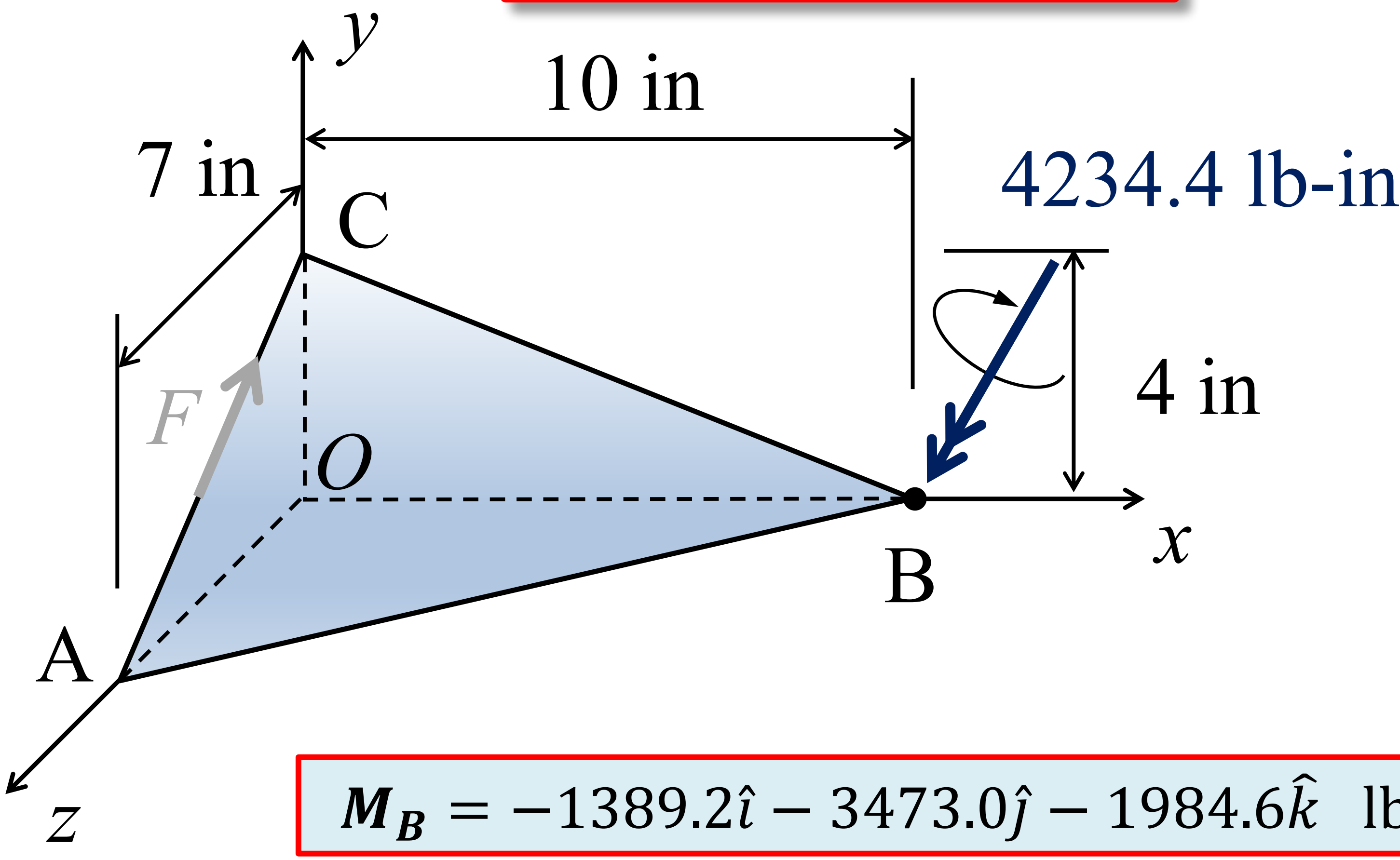
$$\boldsymbol{M_B} = (4)(-347.30)\hat{\imath} + (-10)(198.456)\hat{k} - (-10)(-347.30)\hat{\jmath} \ \text{lb−in}$$

$$\boldsymbol{M_B} = -1389.2\hat{\imath} - 3473.0\hat{\jmath} - 1984.6\hat{k} \ \ \text{lb−in}$$

OK– Same Result

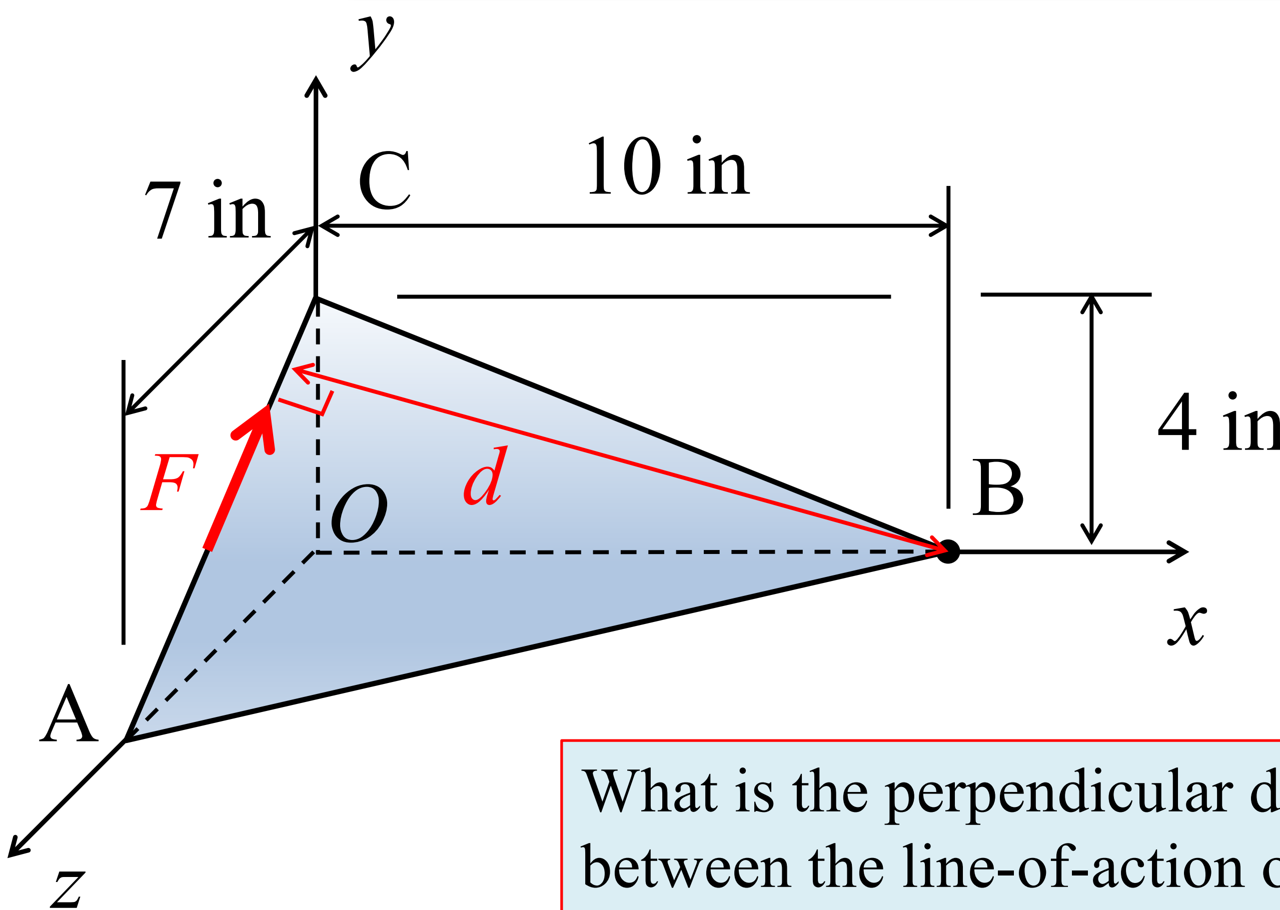
Final Result



$$\boldsymbol{M_B} = -1389.2\hat{\imath} - 3473.0\hat{\jmath} - 1984.6\hat{k} \quad \text{lb–in}$$

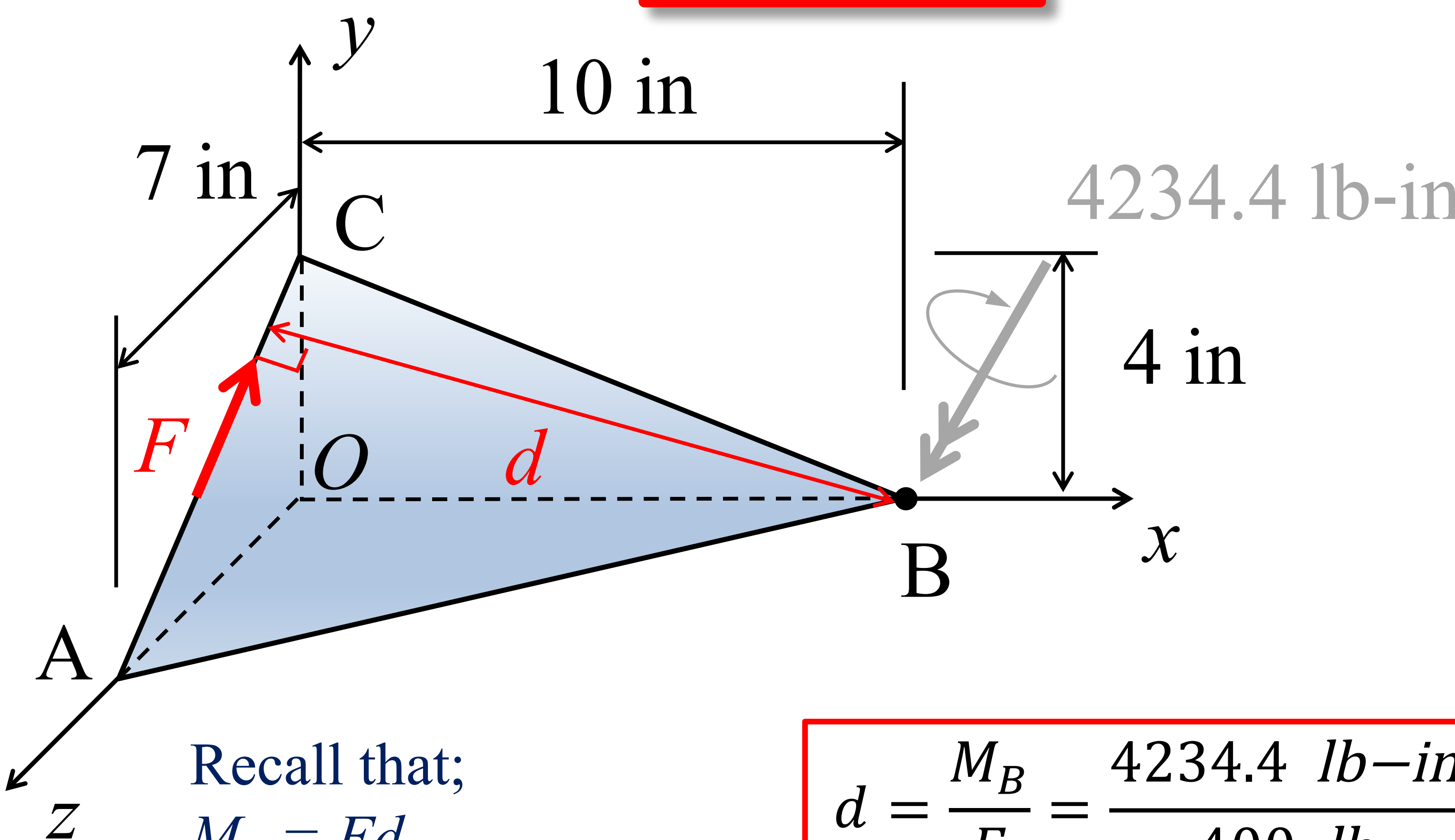
$$M_B = \sqrt{(-1389.2)^2 + (-3473.0)^2 + (-1984.6)^2} = 4234.4 \ \ \text{lb–in}$$

Follow-Up Question



What is the perpendicular distance, *d*, between the line-of-action of ***F*** and point B?

Answer



Recall that;

$M_B = Fd$

and if;

$F = 400$ lb and;

$M_B = 4234.4$ lb-in

$$d = \frac{M_B}{F} = \frac{4234.4\ \ lb{-}in}{400\ \ lb}$$

$$d = 10.59\text{ in}$$